COLORFUL SYMMETRIES

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Pop quiz!

(1) How many different pizzas can be made with 10 optional toppings?
(2) How many ways are there to color a disk partitioned into three equal sectors with one of 2 colors per sector?
(3) How many different 8-bead necklaces can you make out of orange and blue beads?
(4) How many different ways can you color a regular icosahedron with one of n colors on each face if it is allowed to rotate in space?

The combinatorics questions above could be asked by a child, but their solutions range in difficulty from simple high school math to advanced undergraduate math. In particular, each of these questions requires us to think about *symmetry*. A deep understanding of the symmetry involved in combinatorics gives us power to solve all of the above questions and more.

The focus of this article is the last of the questions above, which has a storied history. In 2004, a cryptic URL on a billboard lead to the Google Labs Aptitude Test, a collection of 21 problems, some requiring math and computer science expertise and others rewarding lateral thinking with no correct answer. (One problem asked test-takers to simply fill an empty space with something interesting!) The test included the question,"How many different ways can you color an icosahedron with one of three colors on each face?" We would expect Google to give us fun problems, and the icosahedron coloring question does not disappoint!

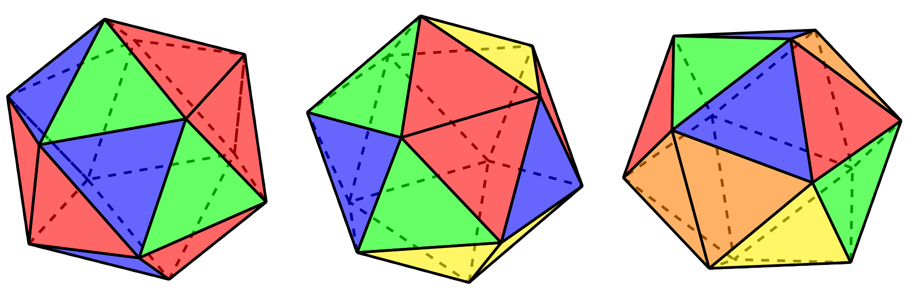


FIGURE 1. How many ways can you color it? Which colorings are the same?

Asking simple questions with difficult answers is easy to do in mathematics. In our solution to this problem, we take a delightful tour through basic geometry, group theory, and combinatorics.

Recall that a regular icosahedron consists of 20 faces, each of which is an equilateral triangle. (For an example, see the MAA logo!) For our problem, we wish to color each face of the icosahedron with one of n colors (see Figure 1). We will permit any face to be colored any of the n colors, regardless of the colors of adjacent faces.

An initial guess for the number of ways to color the icosahedron might be n^{20}, since there are n color choices for each of the 20 faces. In the case of the Google problem where $n = 3$, this gives $3^{20} = 3,486,784,401$ colorings. If we were coloring an icosahedron fixed in space, then this would be the correct answer.

However, consider an icosahedron with 19 green faces and 1 red face. If our icosahedron were fixed in space, then any face could be the red face, so there would be 20 colorings fitting this discription. Instead, we allow our icosahedron to rotate, so we count all of these as only one coloring.

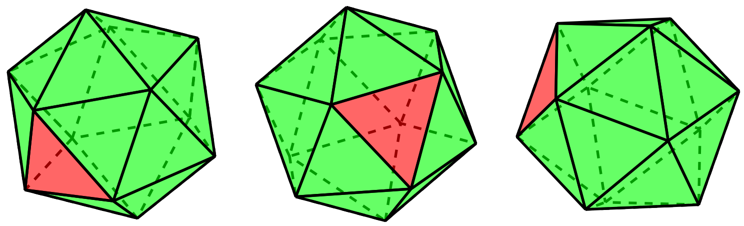


FIGURE 2. Since we allow our icosahedron to rotate, all of these diagrams represent the same coloring.

Each colored icosahedron can be positioned many different ways, so two colored icosahedrons that at first glance appear different may actually be the same if one icosahedron can be rotated so that its coloring matches that of the other. The major task in counting the number of colorings, then, is to determine when two colorings are the same, and this requires us to take the symmetries of the icosahedron into account.

In mathematics it is often useful to look at easier problems when faced with a more difficult one. Determining the symmetries of the icosahedron seems like a daunting task. An easier problem will help us gain some insight.

Starting Simple

Suppose we consider a disk partitioned into three equal sectors, and we want to color each sector one of two colors. How many ways are there to do this? (See Question 2 on the pop quiz.) Figure 3 illustrates all eight *oriented* colorings of this disk. We will name the colorings $x_1, x_2, \ldots, x_8$ and define T to be the set of all oriented colorings. That is, $T = \{x_1, x_2, \ldots, x_8\}$.

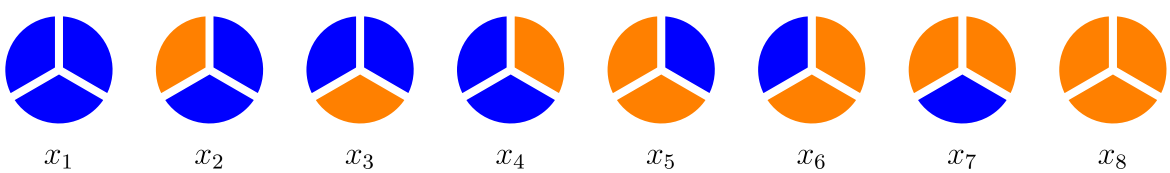


FIGURE 3. Eight oriented colorings of three sectors of a disk with two colors

The number of *oriented* colorings is $|T| = 2^3 = 8$, but by inspection we see that the number of *unoriented* arrangments (up to rotations) is actually 4. Inspecting the colorings works here, but inspection will hardly do when we turn our attention to the icosahedron. Is there a more systematic approach?

In order to more formally account for the rotations, we must consider the symmetries of our disk. By *symmetry* we mean a rotation of the disk that moves each sector to the space occupied by one of the three sectors. Our disk has three symmetries: rotations of 0°, 120°, and 240°. (We include the 0° rotation as the *identity* symmetry.) Together, these symmetries form a *group*, which we will call G. We denote the elements of G by $\pi_0, \pi_{120}, \pi_{240}$, referring to clockwise rotations by 0°, 120°, and 240°, respectively.

Some of the colorings in Figure 3 are really the same coloring, just rotated different ways. If two colorings are the same up to rotation, we say that they are part of the same *equivalence class* of colorings. For example, the colorings x_2, x_3, and x_4 comprise one equivalence class of size 3. It's not a coincidence that the size of G, written $|G|$, is 3 as well.

Each equivalence class of colorings corresponds to *one* coloring of the unoriented disk. Can you see the four equivalence classes in Figure 3?

It would be convenient if each equivalence class contained $|G|$ elements, for then we could simply divide the number of oriented colorings in Figure 3 by $|G|$ to find the number of unoriented colorings. However, in this case we have $2^3 = 8$ oriented colorings and $|G| = 3$, but $8/3$ is not an integer, and thus cannot possibly be the count of unoriented colorings.

The trouble is that in Figure 3, some of the equivalence classes do not contain 3 elements. The equivalence class $\{x_2, x_3, x_4\}$ contains 3 elements, because x_3 and x_4 are rotations of x_2. Likewise, the equivalence class $\{x_5, x_6, x_7\}$ contains 3 elements. However, the "all blue" or "all orange" colorings (x_1 and x_8) are each in their own equivalence class because they are invariant under rotations.

One solution would be to ignore x_3, x_4, x_6 and x_7, but it turns out to be easier to overcount and divide later. Since x_2, x_3, and x_4 are all rotations of the same coloring, we will count all rotations of x_1 and x_8 as well, expanding the equivalence class of each monochrome coloring to make it of size $|G| = 3$. With 3 elements in each equivalence class we have 12 total colorings, and we divide by $|G|$ to obtain $12/3 = 4$ oriented colorings, which is the correct answer.

More formally, we define the function $\phi(x)$ to be the number of symmetries that leave coloring x fixed. Then, for our circle problem, we have

$$\phi(x_1) = \phi(x_8) = 3 \quad \text{and} \quad \phi(x_i) = 1 \text{ for } i = 2, 3, \ldots, 7.$$

We sum all of the $\phi(x_i)$ and divide by the size of the symmetry group G so that the number of colorings N is given by

$$N = \frac{1}{|G|} \sum_{x \in T} \phi(x). \tag{1}$$

For the disk, we have:

$$N = \frac{1}{|G|} \sum_{x \in T} \phi(x) = \frac{1}{3}(3 + 1 + 1 + 1 + 1 + 1 + 1 + 3) = \frac{1}{3}(12) = 4.$$

Figure 4 gives a nice visual representation of our overcounting idea: we want to count all of the highlighted (non-faded) disks. **The basic strategy is to count every orientation of every coloring, and then divide by the size of the group of symmetries.** This is true for this small problem, and it will do the trick for our larger icosahedron problem as well.

So far, we are well-equipped to solve small coloring problems. Our method amounts to working column by column in Figure 4, computing $\phi(x)$ for each oriented coloring x. However, this becomes infeasible for the icosahedron, with its n^{20} oriented colorings. An easier approach is to work row by row in Figure 4, considering first the symmetries of the object to be colored, for there are generally fewer symmetries than oriented colorings. In the next section we explore this row-by-row approach.

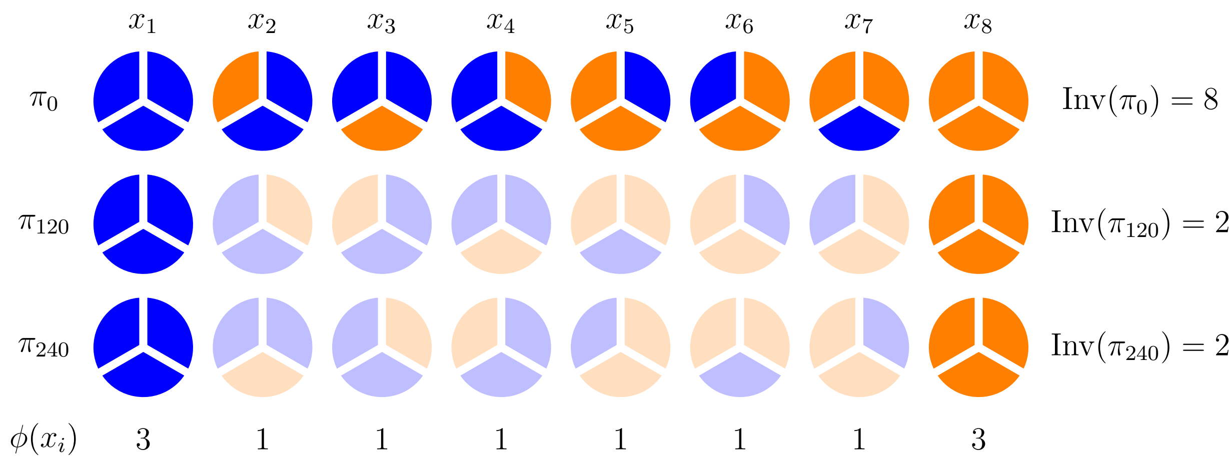


Figure 4. The first row of this diagram is the list of all possible colorings if our disk is fixed in space. In subsequent rows we apply our rotations to these colorings and count the rotated coloring if it is the same as the original. The Inv function gives the number of highlighted disks in a row and the ϕ function gives the number of highlighted disks in a column. Note that each unoriented coloring is counted exactly 3 times.

Burnside's Lemma

Figure 4 displays our overcounted colorings. Each column corresponds to one *oriented* coloring, and each row corresponds to a symmetry. In each column, a highlighted disk appears in rows corresponding to every symmetry that preserves the coloring, while a faded disk appears for each symmetry that does *not* preserve the coloring. For example, coloring x_2 is faded in row π_{120} because $\pi_{120}(x_2) \neq x_2$; that is, $\pi_{120}(x_2)$ is not the same oriented coloring as x_2.

Our overcounted sum $\sum_{x \in T} \phi(x)$ is a count of all the highlighted disks in Figure 4. Previously, we obtained this sum by first counting the number of highlighted disks in each column, and then adding up the column sums. We could obtain the same sum by first counting the number of highlighted disks in each row, and then adding up the row sums.

The number of highlighted disks in each row is really the count of colorings unchanged by each symmetry. This count will be so useful for our purposes that we will define a function, called the *invariant function*, to refer to this count. Specifically, let $\text{Inv}(\pi)$ be the number of colorings left fixed by a symmetry π.

Since we obtain the same sum whether we sum rows or columns of Figure 4, we have:

$$\sum_{x \in T} \phi(x) = \sum_{\pi \in G} \text{Inv}(\pi). \tag{2}$$

To find the number of unoriented colorings, we divide either side of equation (2) by the number of symmetries, $|G|$. That is,

$$N = \frac{1}{|G|} \sum_{\pi \in G} \text{Inv}(\pi). \tag{3}$$

For the disk problem, this becomes:

$$N = \frac{\mathrm{Inv}(\pi_0) + \mathrm{Inv}(\pi_{120}) + \mathrm{Inv}(\pi_{240})}{|G|} = \frac{8+2+2}{3} = 4.$$

Equation (3) is a formulation of a combinatorics result known as *Burnside's Lemma*, and it is what we need to solve the icosahedron problem.

The Icosahedron

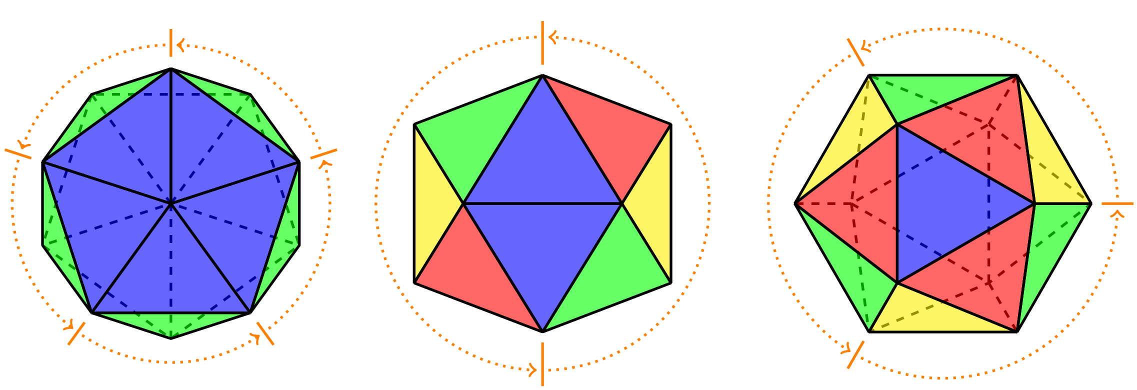
In order to apply equation (3) to the icosahedron coloring problem, we will consider the four types of symmetries of an icosahedron. Let G be the symmetry group of an icosahedron. If you have an icosahedron handy, pull it out and follow along!

Identity symmetry: The simplest "rotation" of an icosahedron simpily leaves it as it is. (If you haven't yet moved your icosahedron, you're applying the identity right now!) Though this rotation is "trivial" in the sense that it does nothing, we have to include it since it's one of the symmetries in G. This symmetry fixes all n^{20} colorings of the icosahedron, so if π_{id} denotes the identity symmetry, then $\mathrm{Inv}(\pi_{\mathrm{id}}) = n^{20}$.

Vertex symmetries: Place your index fingers on opposite vertices and rotate your icosahedron about the axis between the two vertices, as shown in Figure 5(a). Since an icosahedron has six pairs of opposite vertices, there are six choices of axis. For each axis, you can rotate either 72°, 144°, 216°, or 288°. (We don't count the 0° rotation, because that's the identity, which we already considered.) Thus, there are 24 different vertex symmetries.

How many colorings are fixed by each vertex symmetry? Suppose we rotate as depicted in Figure 5(a). The five blue faces form one cycle, because they are permuted by vertex symmetries. The five green faces form another cycle. Similarly, there are two more cycles on the back of the icosahedron, not visible in the figure. There are four cycles, and the faces in each cycle must be colored uniformly in order for the coloring to be invariant under a vertex symmetry. If π_{v} is any vertex symmetry, then $\mathrm{Inv}(\pi_{\mathrm{v}}) = n^4$ colorings are invariant under π_{v}.

Edge symmetries: Now, rotate your icosahedron about an axis connecting midpoints of opposite edges, as in Figure 5(b). An icosahedron has 15 pairs of opposite edges, so there are



(a) Vertex symmetries (b) Edge symmetries (c) Face symmetries

FIGURE 5. Three types of symmetries of an icosahedron. Here, the faces within each *cycle* are colored uniformly.

15 choices of axis. For each axis, there is only one nontrivial symmetry—the 180° rotation, so there are 15 edge symmetries.

To count the colorings fixed by an edge symmetry, we again count cycles. The figure shows four cycles, each colored differently. Likewise, there are four more cycles on the back of the icosahedron. There are also two cycles consisting of faces that are perpendicular to the diagram, and so they don't show up in the figure. There are ten cycles in all, so for any edge symmetry π_e, we have $\text{Inv}(\pi_e) = n^{10}$ colorings invariant under π_e.

Face symmetries: Last, rotate your icosahedron about an axis connecting midpoints of opposite faces, as in Figure 5(c). Ten pairs of opposite faces provide ten axes of rotation, and for each axis we can rotate either 120° or 240° degrees, so we count 20 face symmetries. The figure illustrates four cycles, and there are four more cycles on the back of the icosahedron (note that the cycle colored blue consists of a single face, while other cycles consist of three faces), for a total of eight cycles. If π_f is a face symmetry, then $\text{Inv}(\pi_f) = n^8$.

We have considered one identity symmetry, 24 vertex symmetries, 15 edge symmetries, and 20 face symmetries, for a grand total of 60 symmetries. In fact, these are *all* the symmetries of an icosahedron. Can you think of a more direct way to verify that $|G| = 60$?

We can now compute $\sum_{\pi \in G} \text{Inv}(\pi)$, the sum of the number of colorings fixed by each symmetry. We obtain a polynomial in the number of colors, in which each term corresponds to one type of symmetry:

$$\sum_{\pi \in G} \text{Inv}(\pi) = \underbrace{n^{20}}_{\text{identity}} + \underbrace{24n^4}_{\text{vertex}} + \underbrace{15n^{10}}_{\text{edge}} + \underbrace{20n^8}_{\text{face}} \tag{4}$$

Dividing by $|G|$ as in equation (3), we obtain the number of colorings:

$$N(n) = \frac{n^{20} + 15n^{10} + 20n^8 + 24n^4}{60}. \tag{5}$$

For the Google challenge, $n = 3$ and the number of colorings is $N(3) = 58,130,055$.

Conclusion

Once we knew what we were doing, the computations leading to the solution of the icosahedron problem were painless. We obtained the answer without much trouble because of the helpful machinary we built up along the way. It was useful to start with a simple example with an obvious answer to gain insight into the problem-solving approach. This illustrates a more general strategy that you should appreciate: when you can't do a hard problem, try an easier one and see if it helps you gain intuition about the hard problem.

Coloring problems belong to a fun area of mathematics where geometry, combinatorics, and group theory blend together. The key concept is *symmetry*, which we hope to have illuminated here. These problems provide an excellent launch pad for learning and discussion in the mathematics classroom. As a challenge, you may now want to try quiz question (3), or try to determine the number of colorings of a cube, dodecahedron, or truncated icosahedron!

The authors solved the icosahedron coloring problem while undergraduate students at Messiah College. Brian Bargh (`bbargh@gwmail.gwu.edu`*) is a graduate student at George Washington University, John Chase (*`John_Chase@mcpsmd.org`*) teaches math at Richard Montgomery High School, and Matthew Wright (*`mlwright@ima.umn.edu`*) is a postdoc at the Institute for Mathematics and its Applications. Collectively, the authors can juggle 17 balls.*