Separation of Variables

Section 9.3

1. Follow the steps below to perform Separation of Variables to solve the differential equation

$$\frac{dy}{dx} = \frac{3x + \cos(x)}{12y^2}.$$

(a) First multiply both sides of the equation by dx and $12y^2$. Do you see why the method is called separation of variables?

$$12y^2\,dy = (3x + \cos(x))\,dx$$

(b) Now integrate both sides. The left-hand side you'll integrate with respect to y and the right-hand side with respect to x.

$$\int 12y^2\,dy = \int (3x + \cos(x))\,dx$$

$$4y^3 = \frac{3}{2}x^2 + \sin(x) + C$$

(c) Solve for y in terms of x. Notice you'll have two constants of integration which can be combined into one.

$$y^3 = \frac{3}{8}x^2 + \frac{1}{4}\sin(x) + C$$

(replace $\frac{C}{4}$ with a new C)

$$y = \sqrt[3]{\frac{3}{8}x^2 + \frac{1}{4}\sin(x) + C}$$

2. Using separation of variables find solutions to the following differential equations.

(a) $\dfrac{dy}{dx} = xe^{-y}$

$$e^y\,dy = x\,dx$$

$$\int e^y\,dy = \int x\,dx$$

$$e^y = \frac{1}{2}x^2 + C$$

$$\boxed{y = \ln\left(\frac{1}{2}x^2 + C\right)}$$

(b) $\dfrac{dy}{dx} = \dfrac{\ln(x)}{xy}$

$$y\,dy = \frac{\ln(x)}{x}\,dx$$

$$\int y\,dy = \int \frac{\ln(x)}{x}\,dx$$

$$\frac{1}{2}y^2 = \frac{1}{2}(\ln(x))^2 + C$$

$$y^2 = (\ln(x))^2 + C$$

$$\boxed{y = \pm\sqrt{(\ln(x))^2 + C}}$$

3. Using separation of variables find the solution to each of the following differential equations that satisfies the given initial condition.

(a) $\dfrac{dP}{dt} = \sqrt{Pt}$, $P(0) = 2$

$$\frac{dP}{\sqrt{P}} = \sqrt{t}\,dt$$

$$\int \frac{dP}{\sqrt{P}} = \int \sqrt{t}\,dt$$

$$2P^{1/2} = \frac{2}{3}t^{3/2} + C$$

$$P^{1/2} = \frac{1}{3}t^{3/2} + C$$

$$P(t) = \left(\frac{1}{3}t^{3/2} + C\right)^2$$

$$P(0) = 2 \Rightarrow \left(\frac{1}{3}(0)^{3/2} + C\right)^2 = 2$$

$$C^2 = 2$$

$$C = \sqrt{2}$$

$$\boxed{P(t) = \frac{1}{3}t^{3/2} + \sqrt{2}}$$

(b) $\dfrac{dy}{dx} = \dfrac{xy\sin(x)}{y+1}$, $y(0) = 1$

$$\frac{y+1}{y}\,dy = x\cdot\sin(x)\,dx$$

$$\int \frac{y+1}{y}\,dy = \int x\cdot\sin(x)\,dx$$

$$\int \left(1 + \frac{1}{y}\right)dy = \int x\cdot\sin(x)\,dx$$

integration by parts

$$y + \ln|y| = \sin(x) - x\cdot\cos(x) + C$$

This can't be solved for y.

$$y(0) = 1 \Rightarrow 1 + \ln 1 = \sin(0) + 0\cdot\cos(0) + C$$

$$1 = C$$

$$\boxed{y + \ln|y| = \sin(x) - x\cdot\cos(x) + 1}$$