Dot Product

Section 12.3

One way to multiply two vectors is the dot product:

> If $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, then the **dot product** of $\mathbf{v}$ and $\mathbf{u}$ is:
>
> $$\mathbf{v} \cdot \mathbf{u} = v_1 u_1 + v_2 u_2 + v_3 u_3$$
>
> Note that the dot product of two vectors is a *scalar* (that is, a real number).

1. Find the dot product of the vectors $\mathbf{v} = \langle 3, -1, 4 \rangle$ and $\mathbf{u} = \langle 2, 5, 0 \rangle$.

$$\langle 3, -1, 4 \rangle \cdot \langle 2, 5, 0 \rangle = 3(2) - 1(5) + 4(0) = 6 - 5 = 1$$

2. Let $\mathbf{v} = \langle 2, 4, -1 \rangle$. How does $\mathbf{v} \cdot \mathbf{v}$ relate to $|\mathbf{v}|$? Does your answer hold for *all* vectors?

$$\vec{v} \cdot \vec{v} = 2^2 + 4^2 + (-1)^2 = 21$$

$$|\vec{v}| = \sqrt{2^2 + 4^2 + (-1)^2} = \sqrt{21}$$

In general: $\vec{v} = \langle a, b, c \rangle$

$$\vec{v} \cdot \vec{v} = a^2 + b^2 + c^2 = |\vec{v}|^2$$

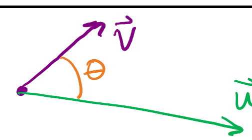
3. If $P = (1, -2, 1)$, $Q = (5, 1, 1)$, and $R = (1, 1, 1)$ are points in $\mathbb{R}^3$, what is $\overrightarrow{PQ} \cdot \overrightarrow{PR}$?

$$\overrightarrow{PQ} = \langle 4, 3, 0 \rangle$$

$$\overrightarrow{PR} = \langle 0, 3, 0 \rangle$$

$$\overrightarrow{PQ} \cdot \overrightarrow{PR} = 4(0) + 3(3) + 0(0) = 9$$

> If θ is the angle between vectors $\mathbf{v}$ and $\mathbf{u}$, then:
>
> $$\mathbf{v} \cdot \mathbf{u} = |\vec{v}|\,|\vec{u}| \cos\theta$$



4. Let $\mathbf{u} = \langle 2, 0, 4 \rangle$ and $\mathbf{v} = \langle -1, 2, 3 \rangle$. If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, find $\cos\theta$.

$$\cos\theta = \frac{\langle 2,0,4 \rangle \cdot \langle -1,2,3 \rangle}{\underbrace{|\langle 2,0,4 \rangle|}_{\text{length of } \vec{u}} \; \underbrace{|\langle -1,2,3 \rangle|}_{\text{length of } \vec{v}}} = \frac{-2 + 0 + 12}{\sqrt{4+0+16}\,\sqrt{1+4+9}} = \frac{10}{\sqrt{20}\,\sqrt{14}} = \frac{10}{2\sqrt{5}\,\sqrt{14}}$$

$$\boxed{\cos\theta = \frac{5}{\sqrt{70}}}$$

$$\theta = \cos^{-1}\left(\frac{5}{\sqrt{70}}\right)$$

5. Find the angle between the vectors $\langle 1, 0, 1\rangle$ and $\mathbf{i}$.

$$\cos\theta = \frac{\langle 1,0,1\rangle \cdot \langle 1,0,0\rangle}{\sqrt{2}\cdot 1} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\theta = \cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4} = 45^\circ$$

6. Two vectors are **orthogonal** (or perpendicular) if the angle between them is $\frac{\pi}{2}$. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then what is $\mathbf{u} \cdot \mathbf{v}$?

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\left(\frac{\pi}{2}\right) = |\vec{u}||\vec{v}|\,0 = 0$$

7. If $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors, how does $\mathbf{u} \cdot \mathbf{v}$ relate to $|\mathbf{u}|$ and $|\mathbf{v}|$?

parallel: $\theta = 0$, so $\cos\theta = 1$

then: $\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|$

"component of $\vec{u}$ in the direction of $\vec{v}$"

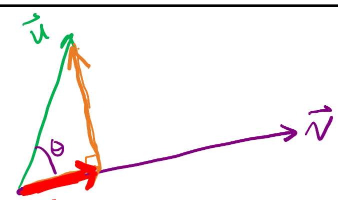
Scalar projection of $\mathbf{u}$ onto $\mathbf{v}$: $\quad \text{comp}_{\mathbf{v}}\mathbf{u} = \dfrac{\vec{v}\cdot\vec{u}}{|\vec{v}|}$

length of the red arrow

Vector projection of $\mathbf{u}$ onto $\mathbf{v}$: $\quad \text{proj}_{\mathbf{v}}\mathbf{u} = \dfrac{\vec{v}\cdot\vec{u}}{|\vec{v}|}\cdot\dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\vec{v}\cdot\vec{u}}{|\vec{v}|^2}\,\vec{v}$

(red vector)

projection of $\vec{u}$ onto $\vec{v}$



projection of $\vec{u}$ onto $\vec{v}$

to be continued...

8. Find the scalar and vector projections of $\mathbf{u} = \langle 2, 3\rangle$ onto $\mathbf{v} = \langle 1, 4\rangle$.

9. Find the scalar and vector projections of $\mathbf{u} = 2\mathbf{j} + \mathbf{k}$ onto $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$.

10. Show that the vector $\mathbf{u} - \text{proj}_{\mathbf{v}}\mathbf{u}$ is orthogonal to $\mathbf{v}$.