Solutions to Practice Problems

Math 126

1. Approximate the function $f(x) = \sqrt{x}$ by a Taylor polynomial of degree 2 at $a = 4$. Use the approximation to estimate $\sqrt{4.1}$.

   First, find two derivatives of $f(x)$:

   $$f'(x) = \frac{1}{2\sqrt{x}}, \qquad f''(x) = \frac{-1}{4x^{3/2}}$$

   Thus, $f(4) = 2$, $f'(4) = \frac{1}{4}$, and $f''(4) = \frac{-1}{32}$.

   By Taylor's Theorem, the Taylor polynomial of degree 2 centered at $a = 4$ is:

   $$T_2(x) = f(4) + f'(4)(x-4) + \frac{f''(4)}{2}(x-4)^2 = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$$

   We use this polynomial to estimate $\sqrt{4.1}$:

   $$\sqrt{4.1} \approx T_2(4.1) = 2 + \frac{1}{4}(4.1-4) - \frac{1}{64}(4.1-4)^2 = 2.0248$$

2. Find the angle between the vectors $\langle 2, 4, 2\rangle$ and $\langle 3, -1, 5\rangle$.

   Let $\mathbf{u} = \langle 2, 4, 2\rangle$ and $\mathbf{v} = \langle 3, -1, 5\rangle$, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Then,

   $$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{12}{\sqrt{24}\sqrt{35}} = \frac{\sqrt{6}}{\sqrt{35}}.$$

   The angle is $\theta = \arccos\left(\frac{\sqrt{6}}{\sqrt{35}}\right) \approx 65.5$ degrees or 1.14 radians.

3. Find an equation of the tangent plane to the surface given by $z = e^x\cos(y) + x + 2$ at the point $(0, 0, 3)$. ss The partial derivatives are $f_x(x, y) = e^x\cos(y) + 1$ and $f_y(x, y) = -e^x\sin(y)$. Thus, $f_x(0, 0) = 2$ and $f_y(0, 0) = 0$. An equation of the tangent plane is

   $$z - 3 = 2(x-0) + 0(y-0) \qquad \text{or} \qquad z = 2x + 3.$$

4. Find the maximum rate of change of the function $f(x, y) = x^2y + \sqrt{xy}$ at the point $(2, 2)$. In which direction does it occur?

   The maximum rate of change is given by the length of the gradient vector, and the direction of maximum rate of change is given by the gradient vector itself. The gradient vector at (x, y) is:

   $$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \left\langle 2xy + \frac{y}{2\sqrt{xy}}, x^2 + \frac{x}{2\sqrt{xy}}\right\rangle$$

   The gradient vector at $(2, 2)$ is:

   $$\nabla f(2, 2) = \left\langle 2(2)(2) + \frac{2}{2\sqrt{(2)(2)}}, 2^2 + \frac{2}{2\sqrt{(2)(2)}}\right\rangle = \langle 8.5, 4.5\rangle$$

   The length of the gradient vector at $(2, 2)$ is:

   $$|\nabla f(2, 2)| = \sqrt{8.5^2 + 4.5^2} = 9.618.$$

   Therefore, the maximum rate of change of f at $(2, 2)$ is 9.618, and it occurs in the direction given by $\langle 8.5, 4.5\rangle$.

5. Find the volume below the surface $z = 2x^2 + y^2 - 2xy$ and above the region $R = [0,2] \times [0,1]$.

The volume is given by:

$$\iint_R (2x^2 + y^2 - 2xy)\,dA = \int_0^2 \int_0^1 (2x^2 + y^2 - 2xy)\,dy\,dx$$

Evaluating the inside integral:

$$\int_0^1 (2x^2 + y^2 - 2xy)\,dy = \left[2x^2 y + \frac{1}{3}y^3 - xy^2\right]_{y=0}^{y=1} = 2x^2 + \frac{1}{3} - x$$

The outside integral is then:

$$\int_0^2 \left(2x^2 + \frac{1}{3} - x\right)\,dx = \left[\frac{2}{3}x^3 + \frac{1}{3}x - \frac{1}{2}x^2\right]_{x=0}^{x=2} = \frac{2}{3}(2)^3 + \frac{1}{3}(2) - \frac{1}{2}(2)^2 = \frac{8}{3}$$

Therefore, the volume is $\frac{8}{3}$.

6. Evaluate the indefinite integral: $\displaystyle\int x^2 \sin(2x)\,dx$

Use integration by parts with $u = x^2$ and $dv = \sin(2x)\,dx$. Thus, $du = 2x\,dx$, $v = -\frac{1}{2}\cos(2x)$, and we have:

$$\int x^2 \sin(2x)\,dx = x^2\left(-\frac{1}{2}\cos(2x)\right) - \int -\frac{1}{2}\cos(2x)\,2x\,dx = -\frac{1}{2}x^2\cos(2x) + \int x\cos(2x)\,dx$$

We must use integration by parts again to evaluate the last integral in the previous line. With $u = x$, $dv = \cos(2x)\,dx$, $du = dx$, and $v = \frac{1}{2}\sin(2x)$, we have:

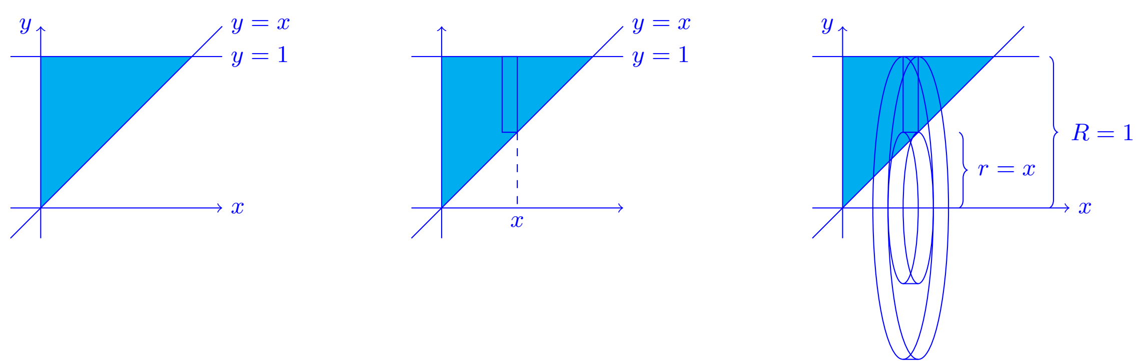
$$\int x\cos(2x)\,dx = x \cdot \frac{1}{2}\sin(2x) - \int \frac{1}{2}\sin(2x)\,dx = \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$$

Combining the previous two lines, we have evaluated the original integral:

$$\int x^2 \sin(2x)\,dx = -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C$$

7. The triangular region bounded by the y-axis, the line $y = x$, and the line $y = 1$ is revolved around the x-axis. Find the volume of the resulting solid.

The triangular region is shown below, in the left diagram. If we were using an integral to find the area of the region, we could let the area element dA, be the vertical rectangle shown in the center diagram below. When finding the volume of revolution, we revolve this rectangle about the x-axis to obtain the volume element, which is the "washer" shown in the right diagram.



The inner radius of the washer is $r = x$. (This is the distance from the x-axis to the line $y = x$.) The outer radius of the washer is $R = 1$. (This is the distance from the x-axis to the line $y = 1$.) The thickness of the washer is dx. Thus, the volume element is the volume of the washer:

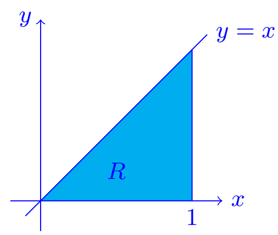
$$dV = \pi R^2\,dx - \pi r^2\,dx = \pi(R^2 - r^2)\,dx = \pi(1 - x^2)\,dx$$

The volume of the solid of revolution is then:

$$V = \int_0^1 dV = \int_0^1 \pi(1 - x^2)\,dx = \pi\left[x - \frac{1}{3}x^3\right]_0^1 = \frac{2\pi}{3}$$

8. Evaluate $\iint_R 3x\,dA$, where R is the region in the xy-plane determined by $0 \le y \le x \le 1$.

The region R is the triangle shown below.



We will integrate with dx on the outside. Note that for any point (x, y) in R, we have $0 \le x \le 1$. For any particular value of x, the corresponding points in R are such that $0 \le y \le x$, which gives the bounds on the inner integral. That is,

$$\iint_R 3x\,dA = \int_0^1 \int_0^x 3x\,dydx.$$

We now evaluate the double integral:

$$\iint_R 3x\,dA = \int_0^1 \int_0^x 3x\,dydx = \int_0^1 3x\left(\int_0^x 1\,dy\right)dx = \int_0^1 3x(x)\,dx = \int_0^1 3x^2\,dx = x^3\big|_0^1 = 1.$$