Bifurcation Plane Lab

Math 230

due Friday, October 20 at 4pm

In this lab you will investigate three families of first-order differential equations that each depend on two parameters, a and r. The goal in each case is to give a sketch of the *bifurcation plane* (or *parameter plane*). The bifurcation plane is a picture in the ar-plane of the regions for which there are different types of phase lines. The curves that separate these regions give the parameter values where bifurcations occur (the *bifurcation curves*).

Here are the three families:

Family 1: $\frac{dy}{dt} = r + ay - y^2$

Family 2: $\frac{dy}{dt} = ry + ay^2$

Family 3: $\frac{dy}{dt} = (y - r)(1 + ay + y^2)$

For each family, do the following:

1. Find all of the equilibrium solutions. These will, of course, depend on a and r. Identify how many equilibrium solutions exist for all pairs (a, r). Classify the equilibrium points.
2. Find the bifurcation curves. To do this, look for values of a and r for which the number or type of equilibria changes. (For example, for what values of a and r do two equilibria come together?)
3. Sketch the bifurcation curves in the ar-plane. Identify the regions in the ar-plane that are bounded by the bifurcation curves. In each different region, draw a representative picture of the phase line for any pair (a, r) in this region.
4. Finally, describe in a sentence or two the bifurcations that occur as (a, r) moves from each region to an adjacent region. Specifically, say how the number or type of equilibrium solutions change across the bifurcation. (For example, a sink and a source merge to form a node.)

Technology

Computational technology is not strictly necessary for this lab, but it may be helpful. For example, you may find it helpful to plot the functions $f(y)$ that appear on the right side of the differential equations above. You could also use Mathematica's `Manipulate` function to see how these plots change as you adjust the values a and r, like this:

```
Manipulate[Plot[r + a*y - y^2, {y, -5, 5}, AxesOrigin -> {0, 0}], {a, -5, 5}, {r,
    -5, 5}]
```

Lab Report

Type your answers to the numbered items above in a document. You may use LaTeX or a word processor (e.g. Microsoft Word). You may draw diagrams by hand. *Do not* turn in a Mathematica file. If you submit your lab electronically, submit a PDF file.

An example of what is expected in this lab is given on the next page for the family $\frac{dy}{dt} = r + ay$.

Grading

Your lab report will be graded out of 40 points, based on the following criteria:

1. **Correctness**: Answers and supporting work are mathematically accurate. In particular, the bifurcation curves have been correctly identified. (20 points)

2. **Completeness**: Assigned questions are answered completely, with appropriate details and justification to support the answers. (8 points)

3. **Clarity**: Explanations are clear and concise. English sentences are used along with equations to explain mathematical reasoning. (6 points)

4. **Presentation**: Work is presented in a typed document that is neat, organized, and easy to read. (6 points)

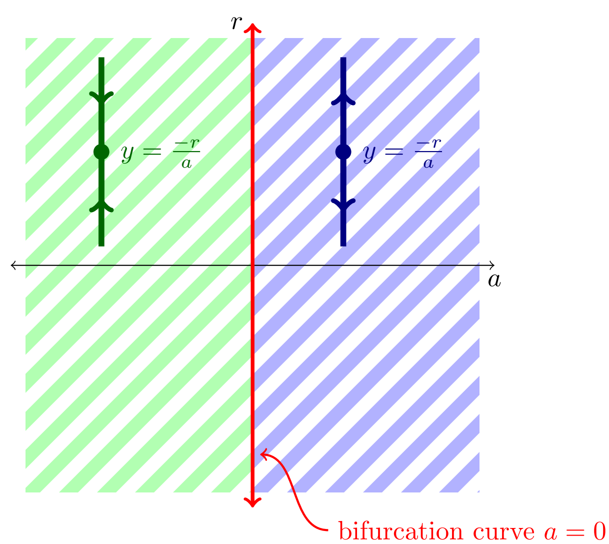
Example

Here we give the bifurcation plane for $\frac{dy}{dt} = r + ay$.

1. We first find all equilibrium solutions. In general, the equilibrium solution is $y = -\frac{r}{a}$, but we need to be careful not to divide by zero. We find the following three cases:
   - If a is nonzero, then there is only one equilibrium solution, which is $y = -\frac{r}{a}$. This equilibrium point is a source if $a > 0$ and a sink if $a < 0$.
   - If $a = 0$ and r is nonzero, then there is no equilibrium solution.
   - If $a = r = 0$, then all points on the phase line are equilibria.

2. From the above, we see that the number of equilibrium solutions for $a = 0$ is different than the number of equilibrium solutions for $a \neq 0$. Thus, $a = 0$ is a bifurcation curve. There are no other bifurcation curves.

3. The bifurcation plane, with representative phase lines in each region, is illustrated here:



4. A bifurcation occurs as a passes through 0. There are three possible ways this can occur:
   - If a passes through zero and $r > 0$, then the equilibrium point moves off to positive infinity when a approaches 0 from the negative side, and then reappears from negative infinity when a is positive. When $a = 0$, there is no equilibrium; all solutions are increasing functions.
   - If a passes through zero and $r < 0$, then the equilibrium point moves off to negative infinity when a approaches 0 from the negative side, and then reappears from positive infinity when a is positive. When $a = 0$, there is no equilibrium; all solutions are decreasing functions.
   - If the parameter pair (a, r) passes through $(0, 0)$, then there is a single equilibrium point when $a \neq 0$, but when $a = 0$ every point on the phase line becomes an equilibrium point.