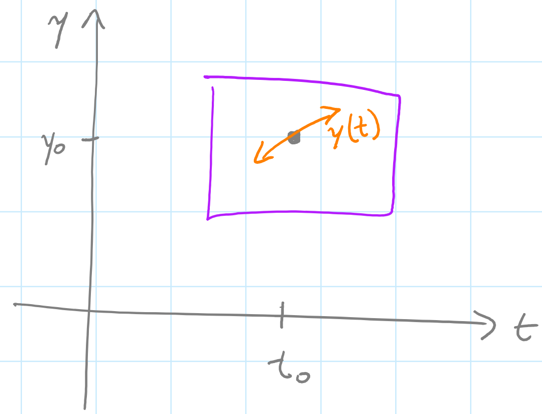
EXISTENCE AND UNIQUENESS

$$\frac{dy}{dt} = f(t,y), \quad y(t_0) = y_0$$

- Does a solution exist?
- Is the solution unique?

EXISTENCE

If $f(t, y)$ is continuous in a rectangle containing (t_0, y_0), then a solution exists through (t_0, y_0).



UNIQUENESS

If $f(t, y)$ and $\frac{\partial f}{\partial y}$ are both continuous in a rectangle containing (t_0, y_0), then the solution is unique near (t_0, y_0).

↳ there is only one solution through (t_0, y_0)

WORKSHEET

Problems and solutions on following pages.

Existence and Uniqueness

Math 230

1. Consider the differential equation $\cos(t)y' - \sin(t)y = 3t\cos(t)$.

   (a) At what points (t, y) does a solution exist?

   First, solve for y': $y' = \tan(t)y + 3t$.

   So $f(t, y) = \tan(t)y + 3t$, which is continuous whenever $t \neq \left(n + \frac{1}{2}\right)\pi$ for integer n. The existence theorem says that a solution exists at these points.

   (b) At what points is the solution unique?

   Since $\frac{\partial f}{\partial y} = \tan(t)$, which is continuous whenever $t \neq \left(n + \frac{1}{2}\right)\pi$ for integer n, the uniqueness theorem says that the solution at these points is unique.

   (c) If a solution $y(t)$ is such that $y(0) = 0$, what is the largest interval on which this solution is unique?

   The solution is unique on the interval $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$.

2. Consider the autonomous differential equation $\frac{dy}{dt} = |y|$.

   (a) What are the equilibrium solutions?

   The only equilibrium solution is $y = 0$.

   (b) For what values of y does a solution exist?

   Since $f(t, y) = |y|$ is continuous everywhere, a solution exists at any point (t, y).

   (c) For what values of y is there a unique solution?

$$\frac{\partial f}{\partial y} = \begin{cases} -1 & \text{if } y < 0 \\ 1 & \text{if } y > 0 \end{cases}$$

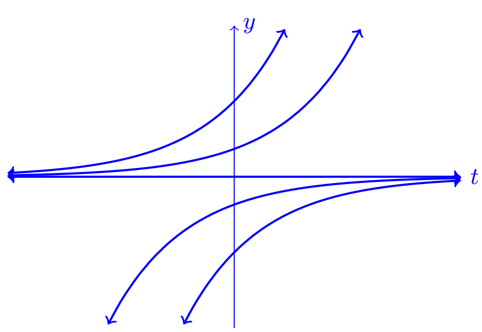
   Since $\frac{\partial f}{\partial y}$ is continuous when $y \neq 0$, the uniqueness theorem guarantees that the solution is unique wherever $y \neq 0$.

   (d) Find all solutions, and sketch the family of solutions. *Hint*: Consider the cases $y > 0$ and $y < 0$ separately, and separate variables. Then consider the case $y = 0$.

   If $y > 0$, then $\frac{dy}{dt} = y$, and we can separate variables to obtain $y = Ke^t$ for some constant $K > 0$.

   If $y < 0$, then $|y| = -y$, so $\frac{dy}{dt} = -y$. Separating variables gives $\frac{dy}{y} = -dt$, and we integrate to obtain $\ln|y| = -t + C$. Since y is negative, $\ln|y| = \ln(-y)$, and we obtain $y = Ke^{-t}$ for some constant $K < 0$.

   Together with the equilibrium solution $y = 0$, we the family of solutions looks like this:



3. Suppose $f(t, y)$ satisfies the hypotheses of the existence and uniqueness theorem for all (t, y). Also suppose that $y_1(t) = 3$, $y_2(t) = 6$, and $y_3(t) = t^2 + 8$ are solutions to $\frac{dy}{dt} = f(t, y)$ for all t. What can you say about solutions satisfying the following initial conditions?

(a) $y(0) = 4$

$3 < y(t) < 6$ for all t

(b) $y(0) = 7$

$6 < y(t) < t^2 + 8$ for all t

(c) $y(0) = 9$

$t^2 + 8 < y(t)$ for all t

4. Consider the autonomous differential equation $\frac{dy}{dt} = \frac{1}{(1+y)^2}$.

(a) For what values of y is there a unique solution?

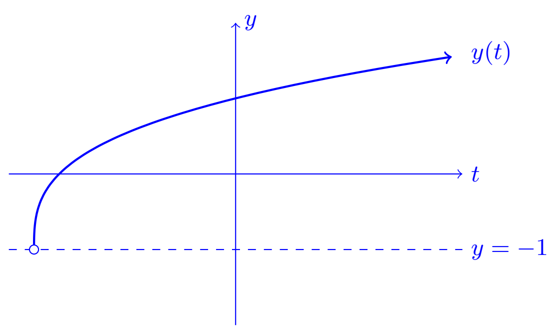
Since $f(t, y) = (1 + y)^{-2}$ and $\frac{\partial f}{\partial y} = -2(1 + y)^{-3}$ are continuous whenever $y \neq -1$, a unique solution exists at any point (t, y) with $y \neq -1$.

(b) Find all solutions.

There are no equilibrium solutions. By separating variables, we find that the general solution is $y(t) = \sqrt[3]{3t + C} - 1$ for some constant C.

(c) If a solution $y(t)$ is such that $y(0) = 1$, what is the largest interval on which this solution exists? Sketch the solution.

The particular solution is $y(t) = \sqrt[3]{3t + 8} - 1$, which exists on the interval $\left(\frac{-8}{3}, \infty\right)$. A sketch of this solution is:



5. Consider the autonomous differential equation $\frac{dy}{dt} = 1 + y^2$.

(a) For what values of y is there a unique solution?

Since $f(t, y) = 1 + y^2$ and $\frac{\partial f}{\partial y} = 2y$ are continuous everywhere, a unique solution exists at any point (t, y).

(b) Find all solutions.

There are no equilibrium solutions. By separating variables, we find that the general solution is $y = \tan(t + C)$ for some constant C.

(c) If a solution $y(t)$ is such that $y(0) = 0$, what is the largest interval on which this solution exists? *Hint*: What is the domain of the solution? Sketch the solution.

The particular solution is $y(t) = \tan(t)$, and the largest interval on which this solution is defined is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.