

CATALAN'S IDENTITY: $F_n^2 - F_{n-r} F_{n+r} = (-1)^{n-r} F_r^2$

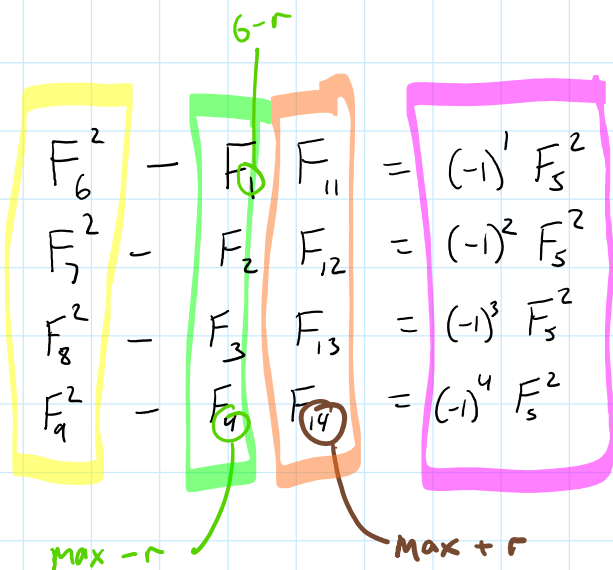
example:
r=5

n=6 := r+1

n=7

n=8

n=9 = max



(careful! don't let this be negative)

Proof of Cassini's Identity: $F_n^2 - F_{n-1} F_{n+1} = (-1)^{n-1}$

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^n$$

$$F_{n+1} F_{n-1} - F_n^2 = \det \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \left(\det \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right)^n = (-1)^n$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

PELL SEQUENCE

$$\frac{x}{y} = \sqrt{2} \text{ has no integer solutions.}$$

Solutions to $x^2 - 2y^2 = \pm 1$ produce $\frac{x}{y}$ that is close to $\sqrt{2}$

Pell's Equation

$$\frac{x^2}{y^2} - 2 = \frac{\pm 1}{y^2} \quad \text{or} \quad \frac{x^2}{y^2} = 2 \pm \frac{1}{y^2}$$

examples: $x=1, y=1$
 $(1)^2 - 2(1)^2 = -1$

So, if y is big, then $\frac{x}{y} \approx \sqrt{2}$

$x=1, y=0$

$x=3, y=2$

$$\frac{3}{2} = 1.5$$

$$\sqrt{2} = 1.414\dots$$

Pell sequence: $P_0 = 0, P_1 = 1, P_{n+1} = 2P_n + P_{n-1}$
 $P_2 = 2, P_3 = 5, P_4 = 12, \dots$

Lucas sequence: $Q_0 = 2, Q_1 = 2, Q_{n+1} = 2Q_n + Q_{n-1}$
 $Q_2 = 6, Q_3 = 14, Q_4 = 34, \dots$

For any n , $x = \frac{Q_n}{2}, y = P_n$ solves Pell's equation.

example: $n=4: x = \frac{Q_4}{2} = \frac{34}{2} = 17, y = 12$

$$(17)^2 - 2(12)^2 = 289 - 2(144) = 1$$

$$\frac{17}{12} = 1.416 \approx \sqrt{2}$$

Look for an identity: $\{P_0, P_3, P_6, P_9, P_{12}, P_{15}\}$
 $\{P_0^3, P_1^3, \dots, P_5^3\}$
 $\{P_0, P_1, \dots, P_5\}$