CATALAN'S IDENTITY: $F_n^2 - F_{n-r}F_{n+r} = (-1)^{n-r}F_r^2$

(careful! don't let this be negative

example:
$r=5$

$6-r$

$n=6: = r+1$ $\quad F_6^2 - F_1 F_{11} = (-1)^1 F_5^2$

$n=7$ $\quad F_7^2 - F_2 F_{12} = (-1)^2 F_5^2$

$n=8$ $\quad F_8^2 - F_3 F_{13} = (-1)^3 F_5^2$

$n=9 = \max$ $\quad F_9^2 - F_4 F_{14} = (-1)^4 F_5^2$

max $-r$ $\qquad$ max $+ r$

Proof of Cassini's Identity: $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n-1}$

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

$$F_{n+1}F_{n-1} - F_n^2 = \det\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1}\end{bmatrix} = \det\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}^n = \left(\det\begin{bmatrix}1&1\\1&0\end{bmatrix}\right)^n = (-1)^n$$

$$\begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix} = \begin{bmatrix}2&1\\1&1\end{bmatrix}$$

$$\begin{bmatrix}1&1\\1&0\end{bmatrix}^3 = \begin{bmatrix}2&1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix} = \begin{bmatrix}3&2\\2&1\end{bmatrix}$$

PELL SEQUENCE

$\frac{x}{y} = \sqrt{2}$ has no integer solutions.

Solutions to $\underbrace{x^2 - 2y^2 = \pm 1}_{\text{Pell's Equation}}$ produce $\frac{x}{y}$ that is close to $\sqrt{2}$

$$\frac{x^2}{y^2} - 2 = \frac{\pm 1}{y^2} \quad \text{or} \quad \frac{x^2}{y^2} = 2 \pm \frac{1}{y^2}$$

So, if y is big, then $\frac{x}{y} \approx \sqrt{2}$

examples: $x=1,\ y=1$

$(1)^2 - 2(1)^2 = -1$

$x=1,\ y=0$

$x=3,\ y=2$ $\qquad \frac{3}{2} = 1.5 \qquad \sqrt{2} = 1.41\ldots$

Pell sequence: $P_0 = 0,\ P_1 = 1,\quad P_{n+1} = 2P_n + P_{n-1}$

$P_2 = 2,\ P_3 = 5,\ P_4 = 12, \ldots$

Lucas sequence: $Q_0 = 2,\ Q_1 = 2,\quad Q_{n+1} = 2Q_n + Q_{n-1}$

$Q_2 = 6,\ Q_3 = 14,\ Q_4 = 34, \ldots$

For any n, $x = \frac{Q_n}{2},\ y = P_n$ solves Pell's equation.

example: $n=4$: $x = \frac{Q_4}{2} = \frac{34}{2} = 17,\quad y = 12$

$(17)^2 - 2(12)^2 = 289 - 2(144) = 1$

$\frac{17}{12} = 1.416 \approx \sqrt{2}$

Look for an identity: $\{P_0, P_3, P_6, P_9, P_{12}, P_{15}\}$

$\{P_0^3, P_1^3, \quad \ldots \quad, P_5^3\}$

$\{P_0, P_1, \ldots, P_5\}$