Mean - Median Map

Start with $x_1, x_2, \dots, x_n$.

Compute: x_{n+1} such that $\frac{x_1 + \cdots + x_n + x_{n+1}}{n+1} = \text{median}(x_1, \dots, x_n)$

Repeat.

THEOREM: The sequence of medians is monotone.

(always increasing or always decreasing)

Start with $S_n = \{x_1, \dots, x_n\}$.

Compute: $x_1 + \cdots + x_n + x_{n+1} = (n+1)\,\text{median}(S_n)$ (1)

$x_1 + \cdots + x_n + x_{n+1} + x_{n+2} = (n+2)\,\text{median}(S_{n+1})$ (2)

Subtract (1) from (2):

$$x_{n+2} = (n+2)\,\text{median}(S_{n+1}) - (n+1)\,\text{median}(S_n)$$

$$x_{n+2} = (n+1)\big(\text{median}(S_{n+1}) - \text{median}(S_n)\big) + \text{median}(S_{n+1})$$

If $\text{median}(S_{n+1}) \geq \text{median}(S_n)$, then is nonnegative, so $x_{n+2} \geq \text{median}(S_{n+1})$. Thus, by definition of median, $\text{median}(S_{n+2}) \geq \text{median}(S_{n+1})$. We have a non-decreasing sequence of medians.

If $\text{median}(S_{n+1}) \leq \text{median}(S_n)$, then similarly, we get a non-increasing sequence of medians.

OBSERVATION

sequence of values:

$$\underbrace{\underbrace{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9}_{\text{median}(S_9)}, x_9}_{\text{median}(S_{10})}, x_9, x_9, \ldots$$

constant

If two consecutive medians are equal, then the sequence of values is constant.

proof follows from the proof above