Conjecture

Ratio of Fibonacci numbers: we observed $\boxed{\frac{F_n}{F_{n-1}} \to \phi = 1.618\ldots}$

proof: Assume that limit $\lim_{n\to\infty} \frac{F_n}{F_{n-1}}$ exists. Let $x = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}$.

Recursive definition: $F_n = F_{n-1} + F_{n-2}$

Divide:

$$\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = 1 + \lim_{n\to\infty} \frac{F_{n-2}}{F_{n-1}}$$

$$\lim_{n\to\infty} \frac{1}{\frac{F_{n-1}}{F_{n-2}}} = \frac{1}{\lim_{n\to\infty} \frac{F_{n-1}}{F_{n-2}}} = \frac{1}{x}$$

$$x = 1 + \frac{1}{x}$$

$$x^2 = x + 1$$

$$x^2 - x - 1 = 0$$

Quadratic Formula: $x = \dfrac{1 \pm \sqrt{1 - 4(-1)}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

Since $x > 0$, it must be that $x = \dfrac{1+\sqrt{5}}{2} = \phi = 1.618\ldots$ ∎

CASSINI'S IDENTITY $F_n^2 - F_{n+1}F_{n-1} = \underline{-1\,(-1)^n} = (1)^{n+1}$

multiply by -1

proof: $F_{n-1}F_{n+1} - F_n^2 = \det\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \det\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \left(\det\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)^n = (-1)^n$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

induction $\begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} & F_n \\ F_{n-1} + F_{n-2} & F_{n-1} \end{bmatrix} = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$

Verification using lists: $F_n^2 - F_{n+1} F_{n-1} = (-1)^{n+1}$

$n=2$ $\quad F_2^2 - F_3 F_1 = -1$

$n=3$ $\quad F_3^2 - F_4 F_2 = 1$

$n=4$ $\quad F_4^2 - F_5 F_3 = -1$

$\vdots \qquad \vdots$

$$F_{m-1}^2 - F_m F_{m-2} = (-1)^{m+1}$$

$$\text{fibSquared} - \underbrace{\text{fibA} * \text{fibB}}_{\text{fibProduct}}$$