We have observed $\frac{F_n}{F_{n-1}} \to 1.618 = \phi$

<u>Conjecture:</u> $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = \phi = 1.618\ldots$

<u>Proof:</u> Assume that $\lim_{n\to\infty} \frac{F_n}{F_{n-1}}$ exists, and let $x = \lim_{n\to\infty} \frac{F_n}{F_{n-1}}$

Recursive definition: $F_n = F_{n-1} + F_{n-2}$

Divide: $\lim_{n\to\infty} \frac{F_n}{F_{n-1}} = 1 + \boxed{\lim_{n\to\infty} \frac{F_{n-2}}{F_{n-1}}}$ $\to \lim_{n\to\infty} \frac{1}{\frac{F_{n-1}}{F_{n-2}}} = \frac{1}{x}$

$$x = 1 + \frac{1}{x}$$

$$x^2 = x + 1$$

$$x^2 - x - 1 = 0$$

Quadratic Formula: $x = \dfrac{1 \pm \sqrt{1-4(-1)}}{2} = \dfrac{1 \pm \sqrt{5}}{2}$

We know $x > 0$, so $x = \dfrac{1+\sqrt{5}}{2} = 1.618\ldots = \phi$ ∎

$$F_n^2 - F_{n+1}F_{n-1} = \underline{(-1)^{n-1}} = (-1)^{n+1}$$

Verification 1: Lists

n = 2 $\quad F_2^2 - F_3 F_1 = -1$

n = 3 $\quad F_3^2 - F_4 F_2 = 1$

n = 4 $\quad F_4^2 - F_5 F_3 = -1$

⋮

$F_{m-1}^2 - F_m F_{m-2} = (-1)^{m-1}$

fibSquared − $\underbrace{\text{fibA} * \text{fibB}}_{\text{fibProduct}}$

Verification 2: modules

Proof: that $F_n^2 - F_{n-1}F_{n+1} = (-1)^{n+1}$

$\cdot(-1)$

$$F_{n-1}F_{n+1} - F_n^2 = \det\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \det\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n = \left(\det\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right)^n = (-1)^n$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} F_3 & F_2 \\ F_2 & F_1 \end{bmatrix}$$

induction: $$\begin{bmatrix} F_n & F_{n-1} \\ F_{n-1} & F_{n-2} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F_n + F_{n-1} & F_n \\ F_{n-1}+F_{n-2} & F_{n-1} \end{bmatrix} = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

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example

```
compute[n_, r_] := Module[ ...

]

Table[compute[n, 2], {n, 1, 100}]
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r=2