Moments of exponential distribution (with $\lambda = 1$) ...

$$E(X^n) = \int_0^\infty x^n e^{-x}\,dx \qquad X \sim Exp(1)$$

... are related to the **GAMMA FUNCTION**:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\,dx$$

For $n = 1, 2, 3, \ldots$, $\Gamma(n) = (n-1)!$

$\Gamma(\alpha)$ extends the factorial to non-integer values!

For $\alpha > 1$, $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$

If $\alpha = n$: $(n-1)! = (n-1)(n-2)!$

$\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $\Gamma(\frac{3}{2}) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{1}{2}\sqrt{\pi}$

GAMMA DISTRIBUTION:

$X \sim \text{Gamma}(\alpha, \beta)$ has pdf $f(x; \alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$

shape scale

If $\alpha = 1$, then gamma distribution is exponential with $\lambda = \frac{1}{\beta}$.

MEAN: $E(X) = \alpha\beta$

VARIANCE: $Var(X) = \alpha\beta^2$

CAUTION: Various parameterizations are commonly used for the gamma distribution. Be very careful when using sources other than our textbook!

R COMMANDS: pgamma$(x, \alpha, \frac{1}{\beta})$ or pgamma$(x, \alpha, \text{scale} = \beta)$ — give $P(X \le x)$ if $X \sim \text{Gamma}(\alpha, \beta)$

qgamma$(p, \alpha, \frac{1}{\beta})$ or qgamma$(p, \alpha, \text{scale} = \beta)$ — give η_p such that $P(X \le \eta_p) = p$

The sum of n independent $Exp(\lambda)$ is a $\text{Gamma}(n, \frac{1}{\lambda})$ random variable.
(more on this in Chapter 4)

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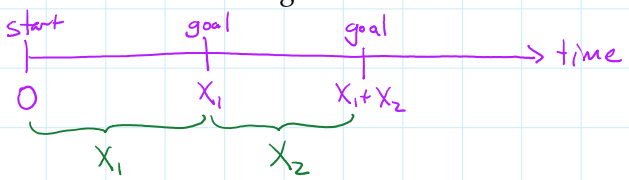
1. Suppose that the time between goals in a hockey game is exponentially distributed with mean 18 minutes (ignore timeouts and stoppages). What is the probability that the second goal occurs less than 30 minutes after the game starts?



$Y = X_1 + X_2$ if $X_1, X_2 \sim Exp\left(\lambda = \frac{1}{18}\right)$, then $Y \sim Gamma\left(\alpha = 2, \beta = 18\right)$.

$P(Y < 30) = \text{pgamma}\left(30, 2, \frac{1}{18}\right) = 0.496$

2. Suppose that a call center receives calls according to a Poisson distribution at a rate of 2 calls per minute. Let Y be the time between 10:00am and the 5th call received after 10:00am.

(a) What are the mean and variance of Y?

Time between calls is $Exp(2)$. Y is the sum of 5 $Exp(2)$ rvs.

Then $Y \sim Gamma\left(5, \frac{1}{2}\right)$.

$E(Y) = \frac{5}{2}$ and $Var(Y) = \alpha\beta^2 = 5\left(\frac{1}{2}\right)^2 = \frac{5}{4}$

(b) What is the pdf of Y?

$$f\left(y; 5, \tfrac{1}{2}\right) = \frac{1}{\left(\frac{1}{2}\right)^5 \Gamma(5)} y^4 e^{-2y} = \frac{4}{3} y^4 e^{-2y}$$

(c) What is $P(Y < 1)$?

$$P(Y < 1) = \int_0^1 \frac{4}{3} y^4 e^{-2y}\, dy = \text{pgamma}(1, 5, 2) = 0.0526$$

3. For large α, the gamma distribution converges to a normal distribution with mean $\alpha\beta$ and variance $\alpha\beta^2$. Investigate this in the case that $\beta = 1$.

(a) Let $X \sim \text{Gamma}(10, 1)$, and compute $P(X \leq 8)$, $P(X \leq 10)$, and $P(X \leq 15)$.

$P(X \leq 8) = 0.2834$ $\quad$ $P(X \leq 15) = 0.9301$

$P(X \leq 10) = 0.5421$ $\quad$ $E(X) = 10(1) = 10$

$P(X \le 10) = 0.5421$

$E(X) = 10(1) = 10$

$Var(X) = 10(1)^2 = 10$, so $\sigma_X = \sqrt{10}$

(b) Let $Z \sim N(10, \sqrt{10})$, and compute $P(Z \le 8)$, $P(Z \le 10)$, and $P(Z \le 15)$.

mean SD

$P(Z \le 8) = 0.2635$

$P(Z \le 15) = 0.9431$

$P(Z \le 10) = \frac{1}{2}$

(c) Now choose a larger value of α and compare $X \sim \text{Gamma}(\alpha, 1)$ with $Z \sim N(\alpha, \sqrt{\alpha})$.