1. An interviewer is given a long list of people that she can interview. When asked, suppose that each person independently agrees to be interviewed with probability 0.45. The interviewer must conduct ten interviews. Let X be the number of people she must ask to be interviewed in order to obtain ten interviews.

   (a) What is the probability that the interviewer will obtain ten interviews by asking no more than 18 people?

   X is negative binomial rv with $p = 0.45$ and $r = 10$.

   $P(X \le 18) = \text{pnbinom}(8, 10, 0.45) = 0.2527$

   (b) What are the expected value and variance of the number of people who *decline* to be interviewed before the interviewer finds ten who agree?

   $E(X-10) = E(X) - 10 = \frac{10}{0.45} - 10 = 12.22$

   $Var(X-10) = Var(X) = \frac{10(1-0.45)}{(0.45)^2} = 27.16$

2. Let $X \sim \text{Geom}(p)$. Find the expected value of $\frac{1}{X}$.

   X has mass function $p(x) = (1-p)^{x-1}p$ for $x = 1, 2, 3, \ldots$

   Then $E\left(\frac{1}{X}\right) = \underbrace{\sum_{x=1}^{\infty} \frac{1}{x}(1-p)^{x-1}p} = \underbrace{\frac{p\ln(p)}{p-1}}$

   To see this, start with the geometric series $\sum_{x=0}^{\infty} r^x = \frac{1}{1-r}$.

   Integrate both sides, and do some algebra.

   Or, use Mathematica or Wolfram Alpha to evaluate the sum.

3. Suppose that $X \sim \text{Exp}(3)$, and let $Y = \lfloor X \rfloor$ denote the largest integer that is less than or equal to X. For example, $\lfloor 2.1 \rfloor = 2$, $\lfloor 5.99 \rfloor = 5$, and $\lfloor 14 \rfloor = 14$.

   (a) Is Y a discrete or continuous random variable?

   Possible values of Y are $0, 1, 2, 3, \ldots$, so Y is discrete.

   (b) Find $P(Y \le 1)$.

   $P(Y \le 1) = P(X < 2) = \int_0^2 3e^{-3x}\,dx = -e^{-3x}\Big|_0^2 = 1 - e^{-6} \approx 0.9975$

(c) Find $P(Y = 2)$.

$$P(Y=2) = P(2 \le X < 3) = \int_2^3 3e^{-3x}\,dx = -e^{-3x}\Big|_2^3 = e^{-6} - e^{-9} \approx 0.0023$$

(d) Can you generalize? What is $P(Y = n)$, for any positive integer n? Is the distribution of Y one of the distributions that we have studied in this course?

$$P(Y=n) = P(n \le X < n+1) = \int_n^{n+1} 3e^{-3x}\,dx = -e^{-3x}\Big|_n^{n+1} = -e^{-3(n+1)} + e^{-3n} = e^{-3n}(1-e^{-3}) = (1-p)^n p,$$

where $p = 1 - e^{-3}$.

This is <u>almost</u> the pmf of a geometric random variable.

In fact, $Y+1$ has a geometric distribution with $p = 1 - e^{-3}$.

4. Let $X \sim \text{Unif}[0,1]$. Compute the nth moment of X in two different ways.

(a) Use the formula $E(X^n) = \int_0^1 x^n dx$.

$$E(X^n) = \int_0^1 x^n\,dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \frac{1}{n+1}$$

(b) Use the moment generating function $M_X(t)$.

$$M_X(t) = \begin{cases} \dfrac{e^t - 1}{t} & \text{if } t \ne 0 \\ 1 & \text{if } t = 0 \end{cases}$$

Recall that $e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!} = 1 + t + \frac{t^2}{2} + \frac{t^3}{3!} + \cdots$.

Thus, as a power series, $M_X(t) = \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!}$.

Reindexing, $M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{(n+1)!} = \sum_{n=0}^{\infty} \frac{1}{n+1} \cdot \frac{t^n}{n!}$.

The coeffient of $\frac{t^n}{n!}$ in the power series is $E(X^n)$, so $E(X^n) = \frac{1}{n+1}$.

5. (a) Give an example of a nonnegative continuous random variable X such that $E(X) < \infty$ and $E(X^2)$ is undefined. (e.g., $E(X^2)$ diverges to ∞.)

One example is X with pdf $f(x) = \frac{3}{2x^{5/2}}$ for $x > 1$.

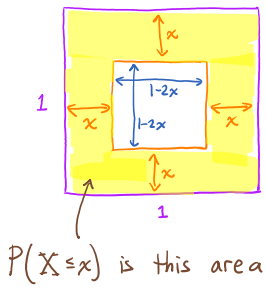
(b) Give an example of a continuous random variable X such that $E(X)$ is undefined. (e.g., $E(X)$ diverges to ∞.)

One example is X with pdf $f(x) = \frac{1}{2x^{3/2}}$ for $x \ge 1$.

Another example is the Cauchy distribution (look it up).

6. Choose a point uniformly at random in a unit square (i.e., a square of side length 1.) Let X be the distance from the point chosen to the nearest edge of the square. Find the pdf of X.

First, find the cdf of X. It's helpful to draw a picture:



$P(X \le x)$ is this area.

If $x \in [0, \frac{1}{2})$, then:

$$F_X(x) = P(X \le x) = 1 - (1-2x)^2 = 4x - 4x^2$$

Thus, the pdf is

$$f_X(x) = \frac{d}{dx} F_X(x) = \frac{d}{dx}(4x - 4x^2) = 4 - 8x \quad \text{for } 0 \le x \le \tfrac{1}{2}.$$

7. Let Y have pdf given by $f_Y(y) = 2(1-y)$ for $0 \le y \le 1$.

(a) Find the density of $U = 2Y - 1$.

$U = g(Y)$, where $g(y) = 2y - 1$.

Since g is monotonic on $0 \le y \le 1$, we can apply the Transformation Theorem.

The inverse of g is $h(u) = \frac{u+1}{2}$, for $-1 \le u \le 1$.

The density of U is then:

$$f_U(u) = f_Y(h(u))\,|h'(u)| = 2\left(1 - \frac{u+1}{2}\right)\left|\frac{1}{2}\right| = \frac{1-u}{2}$$

$$f_U(u) = \frac{1-u}{2} \quad \text{for } -1 \le u \le 1$$

(b) Find the density of $V = Y^2$.

$V = g(Y)$, where $g(y) = y^2$, which is monotonic on $0 \le y \le 1$,

so we can apply the Transformation Theorem.

The inverse of g is $h(v) = \sqrt{v}$ for $0 \le v \le 1$, so the density of V is:

$$f_V(v) = f_Y(h(v))\,|h'(v)| = 2\left(1 - \sqrt{v}\right)\left|\frac{1}{2\sqrt{v}}\right| = \frac{1}{\sqrt{v}} - 1$$

$$f_V(v) = \frac{1}{\sqrt{v}} - 1 \quad \text{for } 0 \le v \le 1$$