

## Final Exam

Math 262: Probability Theory

November 2020

Name: \_\_\_\_\_

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### Instructions:

1. You may refer to your own notes, materials that the professor has posted on the course web site, the textbook, and computational technology (e.g., *R*, *Mathematica*, *Wolfram Alpha*). If you have questions about the exam, you may ask the professor. **Do not consult other sources, people, web sites, etc.**
  2. The St. Olaf Honor Code applies to this exam. Remember the pledge at the end of the exam.
  3. It's acceptable to leave answers in forms such as  $\binom{8}{3}$  or  $0.3 \times 8.2$ .
  4. Notation is as usual:  $P$  denotes probability,  $X$  is a random variable and  $x$  its value,  $E$  is expected value,  $\text{Var}$  is variance, etc.
  5. Read the questions carefully. Check your work.
  6. **The exam is due Thursday, November 19 at 4:00pm.** Please scan and upload your exam to the [Final Exam link on Moodle](#).
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1. (10 points) A fair coin is tossed until heads appears 40 times. Let  $X$  be the number of tosses required.

(a) What is the distribution of  $X$ ? State the values of any parameters necessary to specify this distribution. Use this distribution to find  $P(X \leq 70)$ .

(b) What continuous distribution can be used to approximate the distribution of  $X$ ? State the values of any parameters necessary to specify this distribution. Use this distribution to approximate  $P(X \leq 70)$ .

2. (10 points) Let  $X_1$  and  $X_2$  be independent rvs. The density of  $X_1$  is given by

$$f_{X_1}(x_1) = \begin{cases} 2x_1 & \text{for } 0 \leq x_1 \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

$X_2$  is uniformly distributed on the interval  $[0, 2]$ . Let  $Y_1 = 2X_1 + X_2$  and  $Y_2 = X_1 - X_2$ . Find the joint pdf of  $Y_1$  and  $Y_2$ . Then show that you have found a nonnegative function that integrates to 1 on the appropriate domain.

3. (8 points) Let  $X_1, X_2, \dots, X_5$  be iid random variables with cdf

$$F(x) = \begin{cases} 1 - e^{-x^2} & \text{if } x \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected value of the sample median?

4. (8 points) Random variables  $X$  and  $Y$  have joint density function

$$f(x, y) = \begin{cases} x(2y + 1) & \text{if } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find  $P(X > Y)$ .

5. (12 points) Random variables  $X$  and  $Y$  have joint density function

$$f(x, y) = \begin{cases} 2x + 2y & \text{if } 0 \leq y \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the marginal density function of  $X$ .

(b) Find the conditional density function of  $Y$  given  $X$ .

(c) Find  $P(Y > 0.25 \mid X = 0.5)$ .

6. (10 points) A factory uses machines  $M$ ,  $N$ , and  $P$  to produce certain items. Machine  $M$  produces 40% of the items, machine  $N$  produces 35% of the items, and machine  $P$  produces 25% of the items. Of the items produced by machine  $M$ , 5% are defective. Of the items produced by machine  $N$ , 4% are defective. Of the items produced by machine  $P$ , 3% are defective.

(a) An item is selected at random. What is the probability that it is defective?

(b) If a randomly selected item is defective, what is the probability that it was produced by machine  $M$ ?

7. (10 points) A box contains two blue balls and three yellow balls. Balls are drawn one at a time from the box, randomly and without replacement. Let  $X_1$  be the number of the draw on which the first blue ball is chosen, and let  $X_2$  be the number of the draw on which the second blue ball is chosen. For example, if the balls are drawn *yellow, blue, yellow, blue, yellow*, then  $X_1 = 2$  and  $X_2 = 4$ .

(a) What is the joint pmf of  $X_1$  and  $X_2$ ?

(b) What is  $E(X_2 | X_1 = 2)$ ?

8. (8 points) Let random variable  $X$  have density function

$$f(x) = \begin{cases} 2 - 2x & \text{if } 0 \leq x \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $Y = 3X - 1$ . What is the density function of  $Y$ ?

**St. Olaf Honor Pledge:** I pledge my honor that on this examination I have neither given nor received assistance not explicitly approved by the professor and that I have seen no dishonest work.

Signed: \_\_\_\_\_

I have intentionally not signed the pledge. (Check the box if appropriate.)