1. In how many ways can 12 distinct books be distributed among four (distinct) children so that…

(a) Each child receives three books?

Choose any 3 of 12 books for the first child, then any 3 of the remaining 9 books for the second child, and so on.

$$\binom{12}{3}\binom{9}{3}\binom{6}{3}\binom{3}{3} = 369{,}900 \text{ ways}$$

(b) The two oldest children receive four books each, while the two youngest children receive two books each?

$$\binom{12}{4}\binom{8}{4}\binom{4}{2}\binom{2}{2} = 207{,}900 \text{ ways}$$

2. How many ways can you place 9 identical balls in 4 different boxes?

Select 9 boxes from 4, with replacement, order not important.

$n=4$, $k=9$ number of ways: $\binom{9+4-1}{9} = \frac{12!}{9!\,3!} = 220$

3. How many different dominoes can be formed with the numbers 1, 2, …, 6? How about if the numbers 1, 2, …, 12 are used?

- Choose 2 numbers from 6, with replacement, order unimportant, in $\binom{2+6-1}{2} = \binom{7}{2} = 21$ ways
- For numbers $1, \ldots, 12$: $\binom{2+12-1}{2} = \binom{13}{2} = 78$ ways

4. How many ways can 7 identical jobs be assigned to 10 (distinct) people…

(a) …if no person can do multiple jobs?

Choose 7 of 10 people: $\binom{10}{7} = 120$ ways
(without replacement, order not important)

(b) …if a single person can do multiple jobs?

Now choose 7 of 10 people with replacement, order not important : $\binom{7+10-1}{7} = \binom{16}{7} = 11{,}440$ ways

5. Seven awards are to be distributed to 10 (distinguishable!) mathletes. How many different distributions are possible if

(a) The awards are identical and nobody gets more than one?

Choose 7 out of 10: $\binom{10}{7} = 120$

(b) The awards are different and nobody gets more than one?

Permutations of 7 selected from 10: $\frac{10!}{3!} = 604{,}800$

(c) The awards are identical and anybody can get any number of awards?

Selection with replacement, order doesn't matter.

$$\binom{7+10-1}{7} = \binom{16}{7} = 11{,}440$$

6. Consider the 20 "integer lattice points" (a, b) in the xy-plane given by $0 \le a \le 4$ and $0 \le b \le 3$, with a and b integers. (Draw a little picture.) Suppose you want to walk along the lattice points from (0,0) to (4,3), and the only legal steps are one unit to the right or one unit up.

(a) How many legal paths are there from (0,0) to (4,3)?

Every legal path involves 7 steps, 3 of which are "up."

Choose any 3 of the 7 steps to be "up" in $\binom{7}{3} = 35$ ways.

(b) How many legal paths from (0,0) to (4,3) go through the point (2,2)?

Reasoning as before, there are $\binom{4}{2} = 6$ legal paths from (0,0) to (2,2) and $\binom{3}{1} = 3$ legal paths from (2,2) to (4,3).

Thus, there are $6 \cdot 3 = 18$ paths in all.

7. A box contains 5 red, 6 yellow, and 7 blue balls. The box is stirred and five balls are chosen without replacement. What is the probability that the 5 balls chosen include at least one of each color? Do this in steps:

(a) Let E_1 be the event that *no red ball* is chosen, E_w be the event that *no yellow ball* is chosen, and E_3 be the event that *no blue ball* is chosen. Find the probabilities $P(E_1)$, $P(E_2)$, and $P(E_3)$.

There are $\binom{18}{5}$ ways to choose 5 balls (of any colors).

There are $\binom{13}{5}$ ways to choose 5 balls, none of which are red.

Thus, $P(E_1) = \dfrac{\binom{13}{5}}{\binom{18}{5}} = \dfrac{143}{952} \approx 0.150.$

Similarly, $P(E_2) = \dfrac{\binom{12}{5}}{\binom{18}{5}} = \dfrac{11}{119} \approx 0.092$ and $P(E_3) = \dfrac{\binom{11}{5}}{\binom{18}{5}} = \dfrac{11}{204} \approx 0.054$

(b) Find the probabilities $P(E_1 \cap E_2)$, $P(E_1 \cap E_3)$, $P(E_2 \cap E_3)$, and $P(E_1 \cap E_2 \cap E_3)$.

$E_1 \cap E_2$ is the event that 5 blue balls are chosen, so

$$P(E_1 \cap E_2) = \frac{\binom{7}{5}}{\binom{18}{5}} = \frac{1}{408} \approx 0.002$$

Similarly, $P(E_1 \cap E_3) = \dfrac{\binom{6}{5}}{\binom{18}{5}} = \dfrac{1}{1428} \approx 0.0007$

and $P(E_2 \cap E_3) = \dfrac{\binom{5}{5}}{\binom{18}{5}} = \dfrac{1}{8568} \approx 0.0001.$

Since some balls must be chosen, $P(E_1 \cap E_2 \cap E_3) = 0.$

(c) Use inclusion-exclusion to find $P(E_1 \cup E_2 \cup E_3)$.

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

$$= \frac{359}{1224} \approx 0.293$$

(d) Use the preceding result to answer the original question.

$$P(\text{at least one of each color}) = 1 - P(E_1 \cup E_2 \cup E_3)$$

$$= 1 - \frac{359}{1224} = \frac{865}{1224} \approx 0.707$$

8. Determine how many nonnegative integer solutions satisfy the equation $x_1 + x_2 + x_3 + x_4 = 7$.
   For example, one solution is $x_1 = x_2 = 1$, $x_3 = 0$, $x_4 = 5$, which is different from the solution $x_1 = 1$, $x_2 = 0$, $x_3 = 1$, $x_4 = 5$.
   First rephrase this problem as a selection problem. Is selection with or without replacement? Does order matter?

This is equivalent to the problem of distributing 7 identical balls into 4 distinct boxes: x_1 balls into box 1, x_2 balls into box 2, and so on.

Thus, we are selecting 7 boxes out of 4, with replacement, order not important. We can do this in

$$\binom{7+4-1}{7} = \binom{10}{7} = 120 \text{ ways}$$

Therefore, there are 120 solutions to the equation.

BONUS: (You don't need to know how to do these problems.)

(a) How many ways can 24 students be divided into 4 groups of equal size?

Choose the first group in $\binom{24}{6}$ ways, the second group in $\binom{18}{6}$ ways, the third group in $\binom{12}{6}$ ways, and the fourth group in $\binom{6}{6} = 1$ way.

Since the order of group selection doesn't matter, the number of possible subdivisions is:

$$\frac{\binom{24}{6}\binom{18}{6}\binom{12}{6}\binom{6}{6}}{4!} = 96,197,645,544$$

(b) What is the probability that a randomly chosen arrangement of the letters in MISSISSIPPI contains 4 consecutive Is?

The word MISSISSIPPI has 11 letters, including 4 Ss, 4 Is, and 2 Ps.

The total number of arrangements of these letters is $\dfrac{11!}{4!\,4!\,2!} = 34{,}650$.

To find the number of arrangements with 4 consecutive Is, treat the Is as a single block: $\dfrac{8!}{4!\,2!} = 840$ arrangements.

Thus, the desired probability is $\dfrac{840}{34{,}650} = \dfrac{4}{165} \approx 0.024$.