Definition of Independent Events: $P(A \mid B) = P(A)$

Proposition: A, B independent iff $P(A \cap B) = P(A)P(B)$

7. A sequence of n independent trials are to be performed. Each trial results in a success with probability p and a failure with probability $1-p$. What is the probability that…

(a) …all trials result in successes?

Let A_i be the event that the i^{th} trial results in success.

By independence: $P(n \text{ successes}) = P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1)P(A_2)\cdots P(A_n)$

$= p \cdot p \cdots p = p^n$

(b) …at least one trial results in a success?

Complement: $P(\text{no success}) = P(A_1' \cap A_2' \cap \dots \cap A_n') = (1-p)^n$

Thus: $P(\text{at least 1 success}) = 1 - P(\text{no success}) = 1-(1-p)^n$

(c) …exactly k trials result in successes?

Simplification: $P(\underbrace{A_1 \cap A_2 \cap \dots \cap A_k}_{\text{first } k \text{ successes}} \cap \underbrace{A_{k+1}' \cap A_{k+2}' \cap \dots \cap A_n'}_{\text{remaining } n-k \text{ failures}}) = p^k(1-p)^{n-k}$

How many different ways can I get k successes and $n-k$ failures?

$\binom{n}{k}$ $\qquad \underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}_{n}$ ← choose k to be Heads, the rest Tails

All arrangements occur with prob. $p^k(1-p)^{n-k}$

So: $P(\text{exactly } k \text{ heads}) = \boxed{\binom{n}{k} p^k (1-p)^{n-k}}$

8. If A and B are independent events with positive probability, show that they cannot be mutually exclusive.

If one happens, the other does not

Assume A, B are mutually exclusive.

If A occurs, then B cannot.

So knowledge of A affects probability of B: $P(B|A) = 0 \neq P(B)$

Thus, A and B are dependent.

Pairwise Independence: $P(A \cap B) = P(A)P(B)$

A, B, C are **Mutually Independent** if:

$$P(A \cap B) = P(A)P(B), \quad P(A \cap C) = P(A)P(C)$$

$$P(B \cap C) = P(B)P(C)$$

and $P(A \cap B \cap C) = P(A)P(B)P(C)$