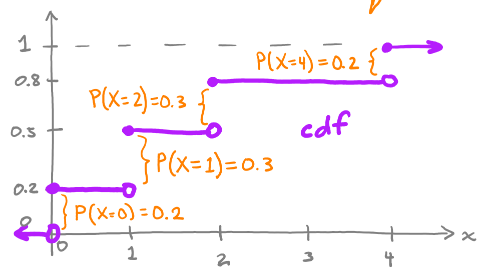
1. The cdf for a random variable X is as follows:

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.2 & 0 \le x < 1 \\ 0.5 & 1 \le x < 2 \\ 0.8 & 2 \le x < 4 \\ 1 & 4 \le x \end{cases}$$

plot



(a) What is $P(X = 2)$?

largest possible value of X that is strictly less than 2

$P(X=2) = F(2) - F(2-) = 0.8 - 0.5 = 0.3$

(b) What is $P(X = 3)$?

$P(X=3) = F(3) - \underbrace{F(3-)}_{} = 0.8 - 0.8 = 0$

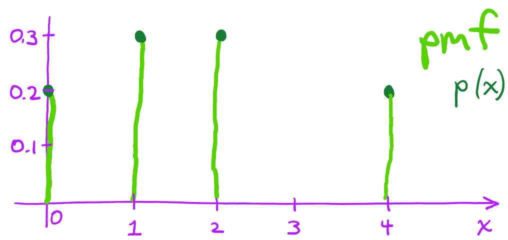
$F(3-) = \lim_{x \to 3^-} F(x)$

(c) What is $P(2.5 \le X)$?

$P(2.5 \le X) = 1 - F(2.5-) = 1 - 0.8 = 0.2$

(d) Sketch the pmf of X.

The nonzero values of $p(x)$ are $p(0) = 0.2$, $p(1) = 0.3$, $p(2) = 0.3$, and $p(4) = 0.2$.



2. Each of the following functions might be the pmf for some random variable X. How can you determine whether a given function is a pmf? Which of these functions is a pmf?

(a) $p(x) = 2 - 3x$ for $x \in \{0,1\}$

This is not a pmf because $p(1) = -1$, but probabilities must be nonnegative.

(b) $p(x) = \frac{x^2}{50}$ for $x \in \{1, 2, \dots, 5\}$

This is not a pmf because $\sum_{x=1}^{5} \frac{x^2}{50} = \frac{1+4+9+16+25}{50} = \frac{55}{50} \neq 1$

(c) $p(x) = \log_{10}\left(\frac{x+1}{x}\right)$ for $x \in \{1, 2, \dots, 9\}$

Since $p(x) \ge 0$ for $x = 1, 2, \dots, 9$ and

and $\sum_{x=1}^{9} \log_{10}\left(\frac{x+1}{x}\right) = \log_{10}\left(\frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \dots \cdot \frac{10}{9}\right) = \log_{10}(10) = 1$, this $\underline{\text{is}}$ a pmf.

(This is the pmf of the distribution known as Benford's Law.)

3. Which of the following properties must hold for any cdf $F(x)$? For each property, either say why it must hold or give a counterexample to show that it might not hold.

(a) $\lim_{b\to-\infty} F(b) = 0$

Yes, since $P(X \le b) \to 0$ as $b \to -\infty$.

(b) $\lim_{b\to\infty} F(b) = 1$

Yes, since $P(X \le b) \to 1$ as $b \to \infty$.

(c) $F(x)$ is continuous

No — consider $F(x)$ in #2 above.

(d) $F(x)$ is nondecreasing; that is, if $a < b$, then $F(a) \le F(b)$

Yes, if $a<b$, then

$$F(a) = P(X \le a) \le P(X \le a) + \underbrace{P(a < X \le b)}_{\text{this is nonnegative}} = P(X \le b) = F(b)$$

(e) $F(b) = 0.5$ for some value b

No — consider the cdf in #2 above.

4. Let X be a random variable with pmf given by $p(4) = 0.3$, $p(5) = 0.2$, $p(8) = 0.3$, $p(10) = 0.2$.

(a) What is the expected value $E(X)$?

$$E(X) = 4(0.3) + 5(0.2) + 8(0.3) + 10(0.2) = 6.6$$

(b) What is $E(X^2)$?

$$E(X) = 4^2(0.3) + 5^2(0.2) + 8^2(0.3) + 10^2(0.2) = 49$$

(c) What is $\mathrm{Var}(X)$? *Hint: use the shortcut formula!*

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2 = 49 - 6.6^2 = 5.44$$

(d) Suppose the random variable is part of a game in which you win $2X - 8$ dollars. Let $Y = 2X - 8$. What is the pmf of Y?

y	0	2	8	12
$p_Y(y)$	0.3	0.2	0.3	0.2

(e) Use the pmf of Y to find $E(Y)$, your expected winnings in this game.

$$E(Y) = 0(0.3) + 2(0.2) + 8(0.3) + 12(0.2) = 5.2$$

(f) Use the pmf of Y to find $E(Y^2)$, and then find $\text{Var}(Y)$.

$$E(Y^2) = 0^2(0.3) + 2^2(0.2) + 8^2(0.3) + 12^2(0.2) = 48.8$$

$$\text{Var}(Y) = E(Y^2) - (E(Y))^2 = 48.8 - (5.2)^2 = 21.76$$

(g) How is $E(Y)$ related to $E(X)$? How is $\text{Var}(Y)$ related to $\text{Var}(X)$?

$E(Y) = 2E(X) - 8$ and $\text{Var}(Y) = 2^2\,\text{Var}(X)$

Expected value is linear, but variance is not!

$$E(aX + b) = aE(X) + b$$

$$E(af(X) + bg(X) + c) = aE(f(X)) + bE(g(X)) + c$$

$$\text{Var}(aX + b) = a^2\,\text{Var}(X)$$

$$\sigma_{aX+b} = |a|\,\sigma_X$$

why? $\text{Var}(aX+b) = E((aX+b)^2) - E(aX+b)^2$

$$= E(a^2X^2 + 2abX + b^2) - (aE(X) + b)^2$$

$$= \cdots = a^2(E(X^2) - E(X)^2) = a\,\text{Var}(X)$$

BONUS: Three balls are randomly selected (without replacement) from an urn containing 20 balls numbered 1 through 20. Let random variable X be the largest of the three selected numbers. What is $P(X = 17)$? What is $P(X \geq 17)$?

Assume that each of the $\binom{20}{3}$ selections are equally likely.

The event $(X \leq x)$ occurs when the three selected balls have numbers less than or equal to x. There are $\binom{x}{3}$ ways to select three such balls. Thus,

$$F(x) = P(X \leq x) = \frac{\binom{3}{x}}{\binom{20}{3}}.$$

We then compute the desired probabilities:

$$P(X = 17) = F(17) - F(16) = \frac{\binom{17}{3}}{\binom{20}{3}} - \frac{\binom{16}{3}}{\binom{20}{3}} = \frac{2}{19} \approx 0.105$$

$$P(X \geq 17) = 1 - F(17) = 1 - \frac{\binom{17}{3}}{\binom{20}{3}} = \frac{29}{57} \approx 0.509$$