From Last Time: Geometric Distribution

$X \sim \text{Geometric}(p)$ means that X counts the number of trials until the first success, where each trial is successful with probability p.

pmf: $P(X=x) = (1-p)^{x-1}p$

geometric tail probability: $P(X>k) = (1-p)^k$

Memoryless Property

For $X \sim \text{Geometric}(p)$ and integers $0 < s < t$,

$$P(X>t \mid X>s) = P(X>t-s).$$

The waiting time until the next success does not depend on how many failures you have already seen.

Moment-Generating Functions

The mgf of discrete random variable X is:

$$M_X(t) = E(e^{tX}) = \sum_x \underbrace{e^{tx}}_{\text{values}} \underbrace{P(X=x)}_{\text{probabilities}}$$

As a power series:

$$M_X(t) = 1 + E(X)t + E(X^2)\frac{t^2}{2} + E(X^3)\frac{t^3}{6} + \cdots$$

To find $E(X^r)$, differentiate $M_X(t)$ r times and evaluate at $t=0$.

EXAMPLE: $X \sim \text{Poisson}(\mu)$

$$P(X=k) = e^{-\mu}\frac{\mu^k}{k!}$$

$$M_X(t) = E(e^{tX}) = \sum_{k=0}^{\infty} e^{tk}\, e^{-\mu}\frac{\mu^k}{k!}$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$= e^{-\mu}\sum_{k=0}^{\infty} e^{tk}\frac{\mu^k}{k!} = e^{-\mu}\sum_{k=0}^{\infty}\frac{(\mu e^t)^k}{k!}$$

same, $x = \mu e^t$

$$= e^{-\mu}\cdot e^{\mu e^t} = e^{\mu(e^t-1)}$$

observe: $M_X(0) = e^{\mu(e^0-1)} = e^0 = 1 = E(X^0)$

$$M_X'(t) = e^{\mu(e^t-1)}(\mu e^t)$$

$$M_X'(0) = e^{\mu(e^0-1)}(\mu e^0) = e^0(\mu\cdot 1) = \mu = E(X^1)$$

$$M_X''(0) = E(X^2)$$

etc.

GEOMETRIC SERIES FORMULAS

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \cdots + ar^n + \cdots = \frac{a}{1-r}$$

$$\sum_{n=0}^{m} ar^n = a + ar + ar^2 + \cdots + ar^m = \frac{a(1-r^{m+1})}{1-r}$$