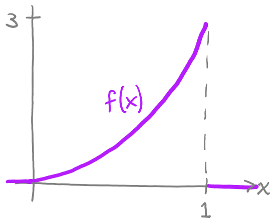
1. **Warm-up:** Let X be a random variable with pdf $f(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$

(a) Sketch $f(x)$. Verify that it really is a pdf.



$f(x) \geq 0$ and

$\int_0^1 f(x)\,dx = \int_0^1 3x^2\,dx = x^3\Big|_0^1 = 1.$

Thus, $f(x)$ is a pdf.

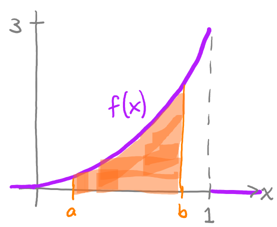
(b) Find $E(X)$ and Var(x).

$$E(X) = \int_0^1 x \cdot 3x^2\,dx = \int_0^1 3x^3\,dx = \frac{3}{4}x^4\Big|_0^1 = \frac{3}{4}$$

$$E(X^2) = \int_0^1 x^2 \cdot 3x^2\,dx = \int_0^1 3x^4\,dx = \frac{3}{5}x^5\Big|_0^1 = \frac{3}{5}$$

$$\text{Var}(X) = E(X^2) - E(X)^2 = \frac{3}{5} - \left(\frac{3}{4}\right)^2 = \frac{3}{80}$$

(c) Find an interval that contains X with probability 0.75.



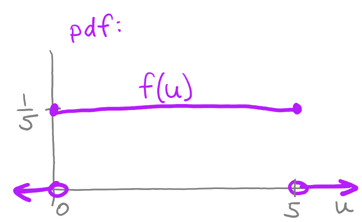
We want to find a and b to make the area of the shaded region equal $\frac{3}{4}$.

One option is to choose $a = 0$. Then:

$$\frac{3}{4} = \int_0^b 3x^2\,dx = x^3\Big|_0^b = b^3$$

So $b = \sqrt[3]{\frac{3}{4}} \approx 0.909.$

2. Let $U \sim$ Unif[0,5]. — so pdf is $f(u) = \frac{1}{5}$ on $[0,5]$



(a) What are the mean and variance of U?

$$E(U) = \int_0^5 u \cdot \frac{1}{5}\,du = \frac{1}{10}u^2\Big|_0^5 = \frac{25}{10} = 2.5 = \frac{B-A}{2}$$

$$E(U^2) = \int_0^5 u^2 \cdot \frac{1}{5}\,du = \frac{1}{15}u^3\Big|_0^5 = \frac{125}{15} = \frac{25}{3}, \quad \text{so} \quad \text{Var}(U) = \frac{25}{3} - \left(\frac{5}{2}\right)^2 = \frac{25}{12} = \frac{(B-A)^2}{12}$$

(b) Let $V = 3U + 2$. What are the mean and variance of V?

$$E(V) = E(3U+2) = 3E(V)+2 = 3(2.5)+2 = 9.5$$

$$Var(V) = Var(3U+2) = 3^2\, Var(U) = 9\left(\frac{25}{12}\right) = \frac{75}{4}$$

RECALL: $Var(aX+b) = a^2\, Var(X)$

(c) What do you think is the distribution of V? *Why?*

If U is uniformly distributed on $[0,5]$, and we rescale this interval linearly to $[2,17]$, then it seems that $V=3U+2$ should be uniformly distributed on $[2,17]$.

3. Let $X \sim \text{Unif}[A,B]$. Show that the mgf of X is $M_X(t) = \begin{cases} \frac{e^{Bt}-e^{At}}{(B-A)t} & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$.

Then use properties of mgfs to verify your answer for 2(c).

$$M_X(t) = E\left(e^{tX}\right) = \int_A^B e^{tx}\frac{1}{B-A}\,dx = \frac{1}{B-A}\cdot\frac{1}{t}e^{tx}\Big|_A^B = \frac{e^{Bt}-e^{At}}{B-A}$$

only if $t \neq 0$

If $t=0$, $\displaystyle\int_A^B e^0 \frac{1}{B-A}\,dx = \frac{B-A}{B-A} = 1.$

Thus: $M_X(t) = \begin{cases} 1 & \text{if } t=0 \\ \frac{e^{Bt}-e^{At}}{(B-A)t} & \text{if } t\neq 0 \end{cases}$ and

The mgf of U from problem 1 is:

$$M_U(t) = \begin{cases} 1 & \text{if } t=0 \\ \frac{e^{5t}-1}{5t} & \text{if } t\neq 0 \end{cases}$$

Since $V=3U+2$, the mgf for V is:

$$M_V(t) = M_{3U+2}(t) = e^{2t}M_U(3t) = e^{2t}\,\frac{e^{5(3t)}-1}{5(3t)} = \frac{e^{17t}-e^{2t}}{15t} \quad \text{if } t\neq 0$$

$$M_V(0) = e^{2t}M_U(0) = 1$$

Note that $M_V(t)$ is the mgf for $\text{Unif}[2,17]$.

Thus, $V \sim \text{Unif}[2,17]$.

4. A stick of length 1 is split at a point U that is uniformly distributed on (0,1).

(a) What is the expected length of the leftmost piece?

The leftmost piece is from 0 to U, so it has length U, and its expected length is $E(U) = \frac{1}{2}$.

(b) What is the expected length of the longest piece?

The two lengths are U and $1-U$, so the length of the longest piece is $\max(U, 1-U)$.

$$E(\max(U, 1-U)) = \int_0^1 \max(u, 1-u) \cdot \underset{\substack{\uparrow \\ \text{pdf of } U}}{1} \, du = \int_0^{\frac{1}{2}} (1-u)\, du + \int_{\frac{1}{2}}^1 u \, du = \left[u - \frac{u^2}{2}\right]_0^{\frac{1}{2}} + \left[\frac{u^2}{2}\right]_{\frac{1}{2}}^1$$

NOTE: $\max(u, 1-u) = \begin{cases} 1-u & \text{if } 0 \le u \le \frac{1}{2}, \\ u & \text{if } \frac{1}{2} < u \le 1. \end{cases}$

$$= \left(\frac{1}{2} - \frac{1}{8}\right) + \left(\frac{1}{2} - \frac{1}{8}\right) = \frac{3}{4}$$

We could also approximate this result via simulation:

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In[18]:= longest[] := Module[{u},
           u = RandomVariate[UniformDistribution[{0, 1}]];
           Return[Max[u, 1 - u]]
         ]

In[21]:= Mean[Table[longest[], 10000]]

Out[21]= 0.747895
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(c) What is the expected length of the piece that contains the point p, $0 \le p \le 1$?

Let $L_p(U) = \begin{cases} 1-U & \text{if } U < p, \\ U & \text{if } U > p. \end{cases}$

Then $$E(L_p(U)) = \int_0^1 L_p(u) \cdot 1 \, du = \int_0^p (1-u)\, du + \int_p^1 u \, du = \left[u - \frac{u^2}{2}\right]_0^p + \left[\frac{u^2}{2}\right]_p^1$$

$$= \left(p - \frac{p^2}{2}\right) + \left(\frac{1}{2} - \frac{p^2}{2}\right) = p - p^2 + \frac{1}{2}$$

Observe that $E(L_p(U))$ is maximized when $p = \frac{1}{2}$.

5. Let X be a random variable that takes on values only between 0 and c. We will show that $\mathrm{Var}(X) \le \frac{c^2}{4}$.

(a) Explain why $E(X^2) \le cE(X)$.

Since $x^2 \le cx$ for $x \in [0,c]$, $E(X^2) = \int_0^c x^2 f(x)\,dx \le \int_0^c cx f(x)\,dx = cE(X)$

(b) Use part (a) to show that $\mathrm{Var}(X) \le c^2[\alpha(1-\alpha)]$, where $\alpha = \frac{E(X)}{c}$.

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 \le cE(X) - E(X)^2 = E(X)(c - E(X))$$

$$= c^2\left[\frac{E(X)}{c} \cdot \frac{c - E(X)}{c}\right] = c^2[\alpha(1-\alpha)]$$

(c) Establish an upper bound on $\alpha(1-\alpha)$ and conclude that $\mathrm{Var}(X) \le \frac{c^2}{4}$.

Note that $\alpha(1-\alpha)$ takes a maximum value of $\frac{1}{4}$ when $\alpha = \frac{1}{2}$.

Therefore, $\mathrm{Var}(X) \le c^2[\alpha(1-\alpha)] \le \frac{c^2}{4}$.