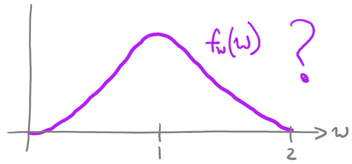
Worksheet Solutions
Math 262 • 22 November 2022

1. Let X and Y be independent uniform variables on $[0,1]$, and let $W = X + Y$.

(a) What do you think the pdf of W will look like? Make a guess. Draw a sketch.

We know $0 \le W \le 2$.
Possibly the pdf of W looks like:



(b) Write down the convolution integral formula for the pdf of W. For what values of x is the integrand nonzero?

$$f_W(w) = \int_{-\infty}^{\infty} \underbrace{f_X(x)}_{} \ \underbrace{f_Y(w-x)}_{} \, dx$$

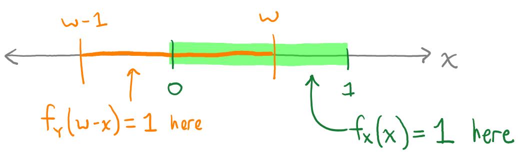
$f_X(x) = 1$ if $\boxed{0 \le x \le 1}$

$f_Y(w-x) = 1$ if $0 \le w-x \le 1$, $\boxed{w-1 \le x \le w}$

If both conditions are true, then integrand is 1; else integrand is 0.

(c) Compute $f_W(w)$ when $0 \le w \le 1$.

If $0 \le w \le 1$:

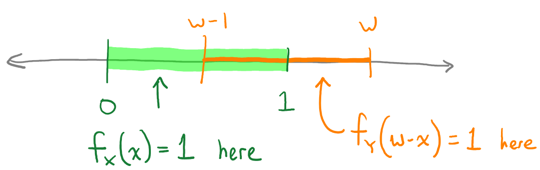


The convolution integrand is nonzero for $0 \le x \le w$:

$$f_W(w) = \int_0^w 1 \, dx = w$$

(d) Compute $f_W(w)$ when $1 \le w \le 2$.

If $1 \le w \le 2$:



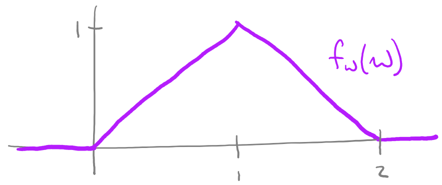
The convolution integrand is nonzero for $w-1 \le x \le 1$:

$$f_W(w) = \int_{w-1}^{1} 1 \, dx = w-(w-1) = 2-w$$

(e) Combine your answers from (c) and (d) to write $f_W(w)$ in piecewise form. Also sketch $f_W(w)$.

The density of W is:

$$f_W(w) = \begin{cases} w & \text{if } 0 \le w \le 1 \\ 2-w & \text{if } 1 \le w \le 2 \\ 0 & \text{otherwise} \end{cases}$$



2. Use convolution to write an integral that gives the pdf of the sum of three independent Unif[0,1] random variables. Use Mathematica to evaluate this integral.

Let $T = X_1 + X_2 + X_3$, $X_i \sim \text{Unif}[0,1]$

$$f_T(t) = \int_0^3 \underbrace{f_X(x)}_{\text{Unif}[0,1]\text{ pdf}} \; f_W(t-x)\, dx$$

$f_W(t-x)$: pdf of X+Y from #1(b)

Mathematica:

Here is the pdf of one Unif[0,1] random variable:

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In[5]:= f1[x_] := Piecewise[{{1, 0 ≤ x ≤ 1}}];
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Here is the pdf of a sum of two Unif[0,1] random variables:

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In[6]:= f2[x_] := Piecewise[{{x, 0 ≤ x ≤ 1}, {2 - x, 1 < x ≤ 2}}]
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The convolution of the previous two density functions gives the density of the sum of three Unif[0,1] random variables.

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In[7]:= f3[x_] := Integrate[f1[t] × f2[x - t], {t, -Infinity, Infinity}]

In[8]:= f3[x]
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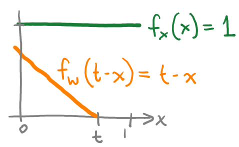
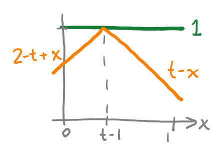
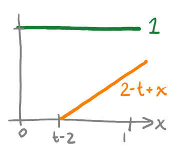
$$\text{Out[8]=} \begin{cases} \frac{x^2}{2} & 0 < x \le 1 \\ \frac{1}{2}\left(-3 + 6x - 2x^2\right) & 1 < x < 2 \\ \frac{1}{2}\left(-3 + 4x - x^2\right) & x == 2 \\ \frac{1}{2}\left(9 - 6x + x^2\right) & 2 < x < 3 \\ 0 & \text{True} \end{cases}$$

If you want to do the integral by hand, here are the details:

$$f_T(t) = \begin{cases} \int_0^t (t-x)\,dx = \frac{1}{2}t^2, & \text{if } 0 \le t < 1 \\ \int_0^{t-1} (2-t+x)\,dx + \int_{t-1}^1 (t-x)\,dx = 3t - t^2 - \frac{3}{2}, & \text{if } 1 \le t < 2 \\ \int_{t-2}^1 (2-t+x)\,dx = \frac{1}{2}t^2 - 3t + \frac{9}{2}, & \text{if } 2 \le t \le 3 \end{cases}$$







3. Let $X_k \sim N(k, 1)$ for $k \in \{1, 2, \dots, m\}$, and suppose all of the X_k are independent.

(a) What is the distribution of $X_1 + X_2 + \cdots + X_m$?

$N(\mu, \sigma)$ has mgf $\exp(\mu t + \sigma^2 t^2/2)$

$$M_{X_k}(t) = \exp\left(kt + t^2/2\right)$$

$$M_{X_1+\cdots+X_m}(t) = \exp\left(t + t^2/2\right)\exp\left(2t + t^2/2\right)\cdots\exp\left(mt + t^2/2\right) = \exp\left((1+2+\cdots+m)t + m\,t^2/2\right) = \exp\left(\tfrac{m(m+1)}{2}t + m\tfrac{t^2}{2}\right)$$

Thus, $X_1 + \cdots + X_m \sim N\left(\frac{m(m+1)}{2}, \sqrt{m}\right)$.

(b) What is the distribution of $X_1 + 2X_2 + \cdots + mX_m$?

$$M_{kX_k}(t) = M_{X_k}(kt) = \exp\left(k^2 t + \tfrac{k^2 t^2}{2}\right)$$

$$M_{X_1+2X_2+\cdots+mX_m}(t) = \exp\left(t + t^2/2\right)\exp\left(4t + 4t^2/2\right)\cdots\exp\left(m^2 t + m^2 t^2/2\right) = \exp\left((1+4+\cdots+m^2)t + (1+4+\cdots+m^2)t^2/2\right)$$

$$= \exp\left(St + St^2/2\right), \quad \text{where} \quad S = 1+4+\cdots+m^2 = \frac{m(m+1)(2m+1)}{6}$$

so the sum is $N(S, \sqrt{S})$.

4. Use moment generating functions to justify the following statements.

(a) The sum of n independent exponential random variables with common parameter λ has a gamma distribution with parameters $\alpha = n$ and $\beta = \frac{1}{\lambda}$.

Let $X_1, X_2, \dots, X_n$ be independent $\text{Exp}(\lambda)$ random variables and $Y = X_1 + X_2 + \cdots + X_n$.

exponential mgf: $M_{X_i}(t) = \dfrac{\lambda}{\lambda - t}$ for $t < \lambda$

Then $M_Y(t) = M_{X_1}(t)M_{X_2}(t)\cdots M_{X_n}(t)$

$$= \left(\frac{\lambda}{\lambda-t}\right)\left(\frac{\lambda}{\lambda-t}\right)\cdots\left(\frac{\lambda}{\lambda-t}\right) = \left(\frac{\lambda}{\lambda-t}\right)^n = \frac{1}{\left(1-\frac{t}{\lambda}\right)^n}$$

mgf of Gamma$\left(\alpha = n, \beta = \frac{1}{\lambda}\right)$

Thus, $Y \sim \text{Gamma}\left(\alpha = n, \beta = \frac{1}{\lambda}\right)$.

(b) The sum of n independent geometric random variables with common parameter p has a negative binomial distribution with parameters $r = n$ and p.

Let $X_1, X_2, \dots, X_n$ be independent $\text{Geom}(p)$ random variables and $Y \sim X_1 + X_2 + \cdots + X_n$.

Geometric mgf: $M_{X_i}(t) = \dfrac{pe^t}{1-(1-p)e^t}$

Then $M_Y(t) = M_{X_1}(t)\, M_{X_2}(t) \cdots M_{X_n}(t) = \left(\dfrac{pet}{1-(1-p)e^t}\right)^n$,

which is the mgf of a negative binomial distribution with parameters $r=n$ and p.