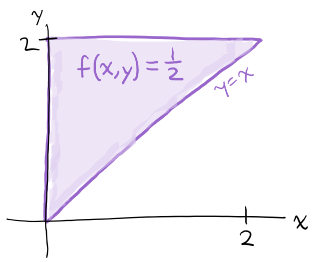
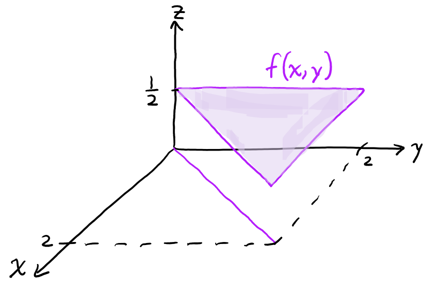
Worksheet Solutions
Math 262 • 29 November 2022

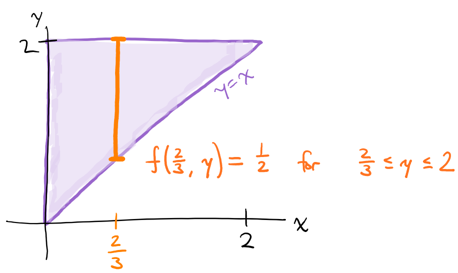
1. Let X and Y have joint density $f(x,y) = \frac{1}{2}$ for $0 \le x \le y \le 2$.

(a) Sketch the joint density of X and Y.





(b) Suppose you know that $X = \frac{2}{3}$. Highlight $x = \frac{2}{3}$ on your sketch in part (a). How can you infer the density of Y given that $X = \frac{2}{3}$?



Since $f(\frac{2}{3}, y)$ is constant, the density of Y given $X = \frac{2}{3}$ is constant for $\frac{2}{3} \le y \le 2$.
That is, if $X = \frac{2}{3}$ then Y is uniform on $[\frac{2}{3}, 2]$,
so $f_{Y|X}(y \mid x = \frac{2}{3}) = \frac{3}{4}$ for $\frac{2}{3} \le y \le 2$.

(c) Let $0 \le x_0 \le 2$ and suppose you know that $X = x_0$. How can you infer the density of Y from $f(x_0, y)$?

Since $f(x_0, y)$ is constant for $x = x_0$, $x_0 \le y \le 2$, Y is uniformly distributed on $[x_0, 2]$.

Thus $f_{Y|X}(x_0, y) = \dfrac{1}{2 - x_0}$ for $x_0 \le y \le 2$.

(d) Find the marginal density $f_X(x)$. Use this to compute the conditional density $f_{Y|X}(y \mid x_0)$. Does this agree with your answer in part (c)?

marginal density: $f_X(x) = \int_x^2 f(x,y)\,dy = \int_x^2 \frac{1}{2}\,dy = \frac{y}{2}\Big|_{y=x}^{y=2} = \dfrac{2-x}{2}$

then: $f_{Y|X}(y \mid x_0) = \dfrac{f(x_0, y)}{f_X(x_0)} = \dfrac{\frac{1}{2}}{\frac{2-x_0}{2}} = \dfrac{1}{2-x_0}$ for $x_0 \le y \le 2$

and this agrees with our answer in part (c).

(e) What is the expected value of Y given that $X = x_0$?

$$E(Y \mid X = x_0) = \int_{x_0}^{2} y \cdot f_{Y|X}(y \mid x_0)\, dy = \int_{x_0}^{2} y \cdot \frac{1}{2-x_0}\, dy = \frac{y^2}{2(2-x_0)}\Bigg|_{y=x_0}^{y=2}$$

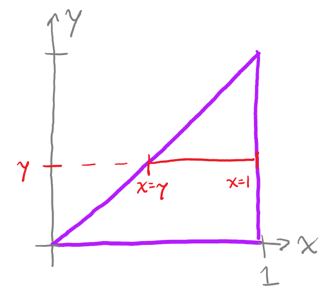
(y: value; $f_{Y|X}(y \mid x_0)$: density)

$$= \frac{4-x_0^2}{2(2-x_0)} = \frac{2+x_0}{2}$$

↑ This is the mean of the Unif $[x_0, 2]$ distribution.

2. The joint pdf of X and Y is $f(x, y) = 3x$, for $0 \le y \le x \le 1$.

(a) What is the conditional distribution of X given $Y = y$?



$$f_Y(y) = \int_y^1 3x\, dx = \frac{3}{2}x^2\Bigg|_{x=y}^{x=1} = \frac{3}{2}(1-y^2) \quad \text{for } 0 \le y \le 1$$

$$f_{X|Y}(x \mid y) = \frac{f(x,y)}{f_Y(y)} = \frac{3x}{\frac{3}{2}(1-y^2)} = \frac{2x}{1-y^2} \qquad 0 \le y \le x \le 1$$

(y: fixed; x: variable)

(b) What is $E(X \mid Y = y)$?

$$E(X \mid Y = y) = \int_y^1 x \cdot \frac{2x}{1-y^2}\, dx = \frac{2}{1-y^2}\int_y^1 x^2\, dx = \frac{2}{1-y^2}\cdot\frac{x^3}{3}\Bigg|_{x=y}^{x=1} = \frac{2}{1-y^2}\left(\frac{1}{3} - \frac{y^3}{3}\right) = \frac{2(1-y^3)}{3(1-y^2)}$$

(c) What is $\mathrm{Var}(X \mid Y = y)$?

$$E(X^2 \mid Y = y) = \int_y^1 x^2 \cdot \frac{2x}{1-y^2}\, dx = \frac{1}{1-y^2}\cdot\frac{x^4}{2}\Bigg|_{x=y}^{x=1} = \frac{1-y^4}{2(1-y^2)} = \frac{1}{2}(1+y^2)$$

$$\mathrm{Var}(X^2 \mid Y = y) = \frac{1}{2}(1+y^2) - \left(\frac{2(1-y^3)}{3(1-y^2)}\right)^2 = \frac{(y-1)^2(y^2+4y+1)}{18(1+y)^2}$$

3. For continuous random variables X and Y, show that $E(E(X \mid Y)) = E(X)$.

inside: conditional expectation
outside: expectation of a function of Y

$$E(X \mid Y=y) = \int_{-\infty}^{\infty} x \cdot f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} x \cdot \frac{f(x,y)}{f_Y(y)}\,dx,$$ which is a function of y

$$E(E(X \mid Y)) = \int_{-\infty}^{\infty} E(X \mid Y=y)\, f_Y(y)\,dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \cdot \frac{f(x,y)}{f_Y(y)}\,dx\, f_Y(y)\,dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f(x,y)\,dy\,dx = \int_{-\infty}^{\infty} x \left(\int_{-\infty}^{\infty} f(x,y)\,dy\right) dx$$

($\int_{-\infty}^{\infty} f(x,y)\,dy = f_X(x)$)

$$= \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = E(X)$$

4. The number of eggs N found in a nest of a certain species of turtle has a Poisson distribution with mean λ. Each egg has a probability p of being viable, and this event is independent from egg to egg. Find the mean and variance of the number of viable eggs per nest.

rvs: $N \sim \text{Poisson}(\lambda)$, X = number of viable eggs $\sim \text{Bin}(N, p)$

mean: $E(X) = E(E(X \mid N)) = E(Np) = p\,E(N) = p\lambda$

$E(X \mid N) = Np$

variance: $Var(X) = Var(E(X \mid N)) + E(Var(X \mid N)) = Var(Np) + E(Np(1-p))$

$$= p^2\, Var(N) + p(1-p)\,E(N) = p^2\lambda + p(1-p)\lambda = p^2\lambda - p^2\lambda + p\lambda = p\lambda$$

5. Simulate 10,000 averages, each of k samples from a Unif[0,1] distribution. Make a histogram of the 10,000 averages. Start with $k = 1$ and then try larger values of k. How does the shape of the histogram depend on k?

We will do this next time.

6. Repeat the previous simulation, but now replace Unif[0,1] with a different distribution of your choice. What is the shape of the histogram? How does it depend on k?

Next time...

BONUS: If X and Y are independent binomial random variables with identical parameters n and p, calculate the conditional expected value of X given that $X + Y = m$.

First, compute the conditional pmf of X given that $X+Y=m$.

$$P(X=k \mid X+Y=m) = \frac{P(X=k \text{ and } X+Y=m)}{P(X+Y=m)} = \frac{P(X=k)\,P(Y=m-k)}{P(X+Y=m)}$$

$$= \frac{\binom{n}{k} p^k (1-p)^{n-k} \cdot \binom{n}{m-k} p^{m-k} (1-p)^{n-m+k}}{\binom{2n}{m} p^m (1-p)^{2n-m}}$$

[For the denominator, note that $X+Y \sim \text{Bin}(2n, p)$]

$$= \frac{\binom{n}{k}\binom{n}{m-k}}{\binom{2n}{m}}$$

This is a hypergeometric probability: the probability of k successes in a sample of size m from a population with n successes and n failures.

So the conditional distribution of X, given that $X+Y=m$, is hypergeometric, and its mean is $E(X \mid X+Y=m) = \frac{m}{2}$.