1. The probability that Prisca studies for a test is 0.8. The probability that she studies and passes the test is 0.7. If Prisca studies, what is the probability that she passes the test?

Events: A: Prisca studies B: Prisca passes

Conditional Probability $\rightarrow$ $P(B|A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{0.7}{0.8} = \dfrac{7}{8}$

2. A machine produces parts, 10% of which are defective. An inspector is able to remove 95% of the defective parts. What is the probability that a part is defective *and* removed by the inspector?

Events: D: part is defective; R: part is removed

We know: $P(D) = 0.1$, $P(R|D) = 0.95$

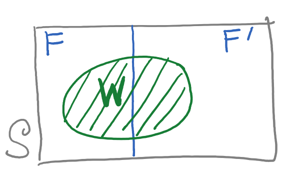
$$\underbrace{P(R \cap D) = P(R|D)\,P(D)}_{\text{Multiplication Rule}} = (0.95)(0.1) = 0.095$$

3. A soccer team wins 60% of its games when it scores the first goal, and 30% of its games when the opposing team scores first. If the team scores first in 40% of its games, what percent of its games does it win?

Events: W: win, F: score first

We know: $P(F) = 0.4$, $P(W|F) = 0.6$, $P(W|F') = 0.3$

Find $P(W)$:



$$\begin{aligned} P(W) &= P(W \cap F) + P(W \cap F') \\ &= P(W|F)P(F) + P(W|F')P(F') \\ &= (0.6)(0.4) + (0.3)(0.6) \\ &= 0.42 \end{aligned}$$

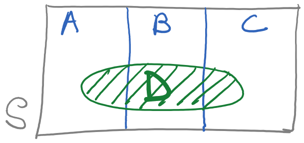
4. A factory uses 3 machines to produce certain items. Machine A produces 50% of the items, 6% of which are defective. Machine B produces 30% of the items, 4% of which are defective. Machine C produces 20% of the items, 3% of which are defective.

(a) What is the probability that a randomly-selected item is defective?

Events: A: machine A, B: machine B, C: machine C
D: defective

Find $P(D)$:

$$P(D) = P(D \mid A)\,P(A) + P(D \mid B)\,P(B) + P(D \mid C)\,P(C)$$
$$= (0.06)(0.5) + (0.04)(0.3) + (0.03)(0.2)$$
$$= 0.048$$



(b) If an item is defective, what is the probability that it was produced by Machine A?

$$P(A \mid D) = \frac{P(A \cap D)}{P(D)} = \frac{P(D \mid A)\,P(A)}{P(D)} = \frac{(0.06)(0.5)}{0.048} = \frac{5}{8}$$

Bayes' theorem!

5. Suppose that a patient is tested for a disease. Let A be the event that the test is positive, and let D be the event that the patient actually has the disease. Further suppose that:

$P(A \mid D) = 0.99$	(*sensitivity*: probability of a positive test if the patient has the disease)
$P(A' \mid D') = 0.99$	(*specificity*: probability of a negative test if the patient doesn't have the disease)

(a) *Rare disease*: If $P(D) = 0.01$, what is the probability that a patient who tests positive actually has the disease?

$$P(D \mid A) = \frac{P(A \mid D)\,P(D)}{P(A)} = \frac{P(A \mid D)\,P(D)}{P(A \mid D)P(D) + P(A \mid D')P(D')} = \frac{(0.99)(0.01)}{(0.99)(0.01) + (0.01)(0.99)} = \frac{1}{2}$$

Imagine testing 1000 people: 990 without the disease and 10 with the disease

- Of the 10 with the disease, all test positive
- Of the 990 without disease, about 10 are false positives

$\frac{10}{20}$ positive tests are true positives

(b) *Common disease*: If $P(D) = 0.1$, what is the probability that a patient who tests positive actually has the disease?

$$P(D \mid A) = \frac{(0.99)(0.1)}{(0.99)(0.1) + (0.01)(0.9)} = 0.917$$

Imagine testing 1000 people: 900 without the disease and 100 with the disease

- Of the 100 with the disease, about 99 test positive
- Of the 900 without the disease, about 9 are false positives

$\frac{99}{108}$ positive tests are true positives

BONUS: Box 1 contains 5 red balls and box 2 contains 5 blue balls. Balls are randomly removed in the following manner: after each removal from box 1, a ball is taken from box 2 (if box 2 has any balls) and placed in box 1. This process continues until all balls have been removed (so ten removals total). What is the probability that the final ball removed from box 1 is red?

This problem is tricky! It won't be on the exam.

Number the red balls 1, 2, 3, 4, 5. Let R_i be the event that red ball i is the final ball selected.

Now focus on a particular red ball, say ball 1. Let N_j be the event that this ball is <u>not</u> removed on the j^{th} draw from box 1, for $j \in \{1,2,3,4,5\}$.

Then:

$$\begin{aligned} P(R_1) &= P(N_1 \cap N_2 \cap N_3 \cap N_4 \cap N_5 \cap R_1) \\ &= P(N_1) \cdot P(N_2 \mid N_1) \cdot P(N_3 \mid N_1 \cap N_2) \cdot P(N_4 \mid N_1 \cap N_2 \cap N_3) \\ &\quad \cdot P(N_5 \mid N_1 \cap N_2 \cap N_3 \cap N_4) \cdot P(R_1 \mid N_1 \cap N_2 \cap N_3 \cap N_4 \cap N_5) \\ &= \underbrace{\frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5} \cdot \frac{4}{5}}_{\text{The first 5 draws occur when there are 5 balls in Box 1.}} \cdot \frac{1}{5} = \left(\frac{4}{5}\right)^5 \cdot \frac{1}{5} \end{aligned}$$

Probability that red ball 1 is selected <u>last</u> among the last 5 draws from box 1 (after box 2 is empty).

Similarly, $P(R_i) = \left(\frac{4}{5}\right)^5 \cdot \frac{1}{5}$ for $i \in \{2,3,4,5\}$.

Since the events $R_1, R_2, \ldots, R_5$ are disjoint:

$$\begin{aligned} P(\text{some red ball is selected last}) &= P(R_1) + P(R_2) + \cdots + P(R_5) \\ &= 5 \cdot \left(\frac{4}{5}\right)^5 \cdot \frac{1}{5} = \boxed{\left(\frac{4}{5}\right)^5} \end{aligned}$$

[**GENERALIZATION:** If we start with n red balls and n blue balls, the probability that a red ball is selected last is $\left(\frac{n-1}{n}\right)^n$, which converges to $\frac{1}{e}$ as $n \to \infty$.]