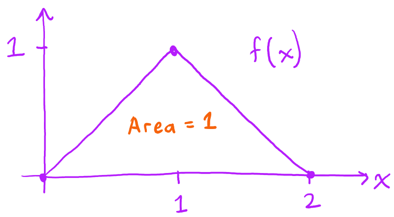
1. Let X be a continuous random variable with pdf

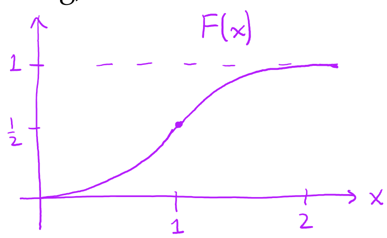
$$f(x) = \begin{cases} x & 0 \le x \le 1, \\ 2 - x & 1 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Sketch the pdf of X.



(b) Without computing anything, sketch cdf of X.

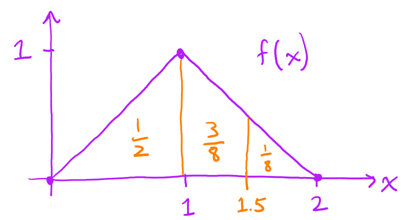
The cdf must look something like this:



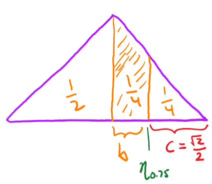
(c) What is $P(X < 1.5)$?

Find the area under the pdf left of $x = 1.5$.

$$P(X < 1.5) = \int_0^{1.5} f(x)\,dx = \frac{7}{8}$$



(d) Find a value $\eta_{0.75}$ such that $P\big(X \le \eta_{0.75}\big) = 0.75$.



Find c such that c [triangle with sides c, c and angle $45°$] has area $\frac{1}{4}$.

$\frac{c^2}{2} = \frac{1}{4}$

$c^2 = \frac{1}{2}$

$c = \frac{\sqrt{2}}{2}$

so $\eta_{0.75} = 2 - \frac{\sqrt{2}}{2}$

≈ 1.293

(e) Give a formula for the cdf of X.

If $0 \le x \le 1$, then $\underset{\text{cdf: } P(X \le x)}{F(x)} = \int_0^x \underset{\text{pdf}}{y}\,dy = \frac{1}{2}y\Big|_{y=0}^{y=x} = \frac{1}{2}x^2$

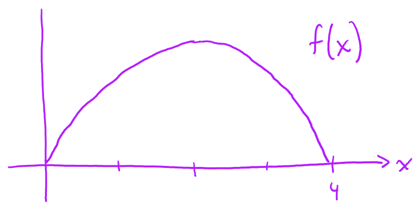
If $1 \le x \le 2$, then $F(x) = \underbrace{\frac{1}{2}}_{P(X \le 1)} + \underbrace{\int_1^x (2-y)\,dy}_{P(1 \le X \le x)} = \frac{1}{2} + \left[2y - \frac{1}{2}y^2\right]_{y=1}^{y=x} = 2x - \frac{x^2}{2} - 1$

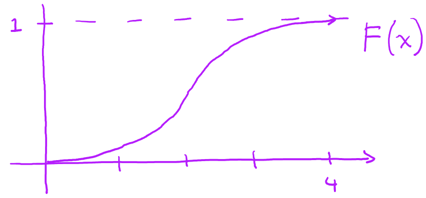
Thus:

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{2}x^2 & \text{if } 0 \le x \le 1 \\ 2x - \frac{x^2}{2} - 1 & \text{if } 1 < x \le 2 \\ 1 & \text{if } 2 < x \end{cases}$$

2. Suppose that a continuous random variable X has pdf $f(x) = kx(4-x)$ for $0 \le x \le 4$, and $f(x) = 0$ otherwise.

(a) Sketch the pdf of X. Then, without computing anything, sketch the cdf of X.





(b) What is the value of k?

Remember that the pdf must integrate to 1:

$$\int_0^4 kx(4-x)\,dx = \frac{32}{3}k = 1, \quad \text{so } k = \frac{3}{32}.$$

(c) Find $P(X > 3 \text{ or } X < 1)$.

$$P(X > 3 \text{ or } X < 1) = \int_0^1 f(x)\,dx + \int_3^4 f(x)\,dx = 2\int_0^1 \frac{3}{32}x(4-x)$$

by symmetry of f(x) about x=2

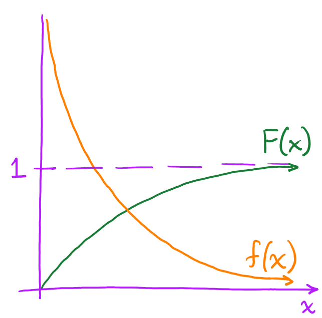
$$= \frac{3}{16}\left[2x^2 - \frac{x^3}{3}\right]_0^1 = \frac{3}{16}\left(2 - \frac{1}{3}\right) = \frac{5}{16}$$

3. Suppose that the cdf of a random variable X is $F(x) = 1 - e^{-5x}$ for $x > 0$, and $F(x) = 0$ otherwise.

(a) What is the pdf of X? Sketch both the pdf and the cdf.

Differentiate the cdf to find the pdf:

$$f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\left(1 - e^{-5x}\right) = 5e^{-5x} \quad \text{for } x > 0.$$



(b) What is $P\left(\frac{1}{4} < X < \frac{1}{3}\right)$? Can you get this from *either* the cdf or the pdf?

cdf: $P\left(\frac{1}{4} < X < \frac{1}{3}\right) = F\left(\frac{1}{3}\right) - F\left(\frac{1}{4}\right) = \left(1 - e^{-5/3}\right) - \left(1 - e^{-5/4}\right) = e^{-5/4} - e^{-5/3} \approx 0.098$

pdf: $P\left(\frac{1}{4} < X < \frac{1}{3}\right) = \int_{\frac{1}{4}}^{\frac{1}{3}} f(x)\,dx = \nearrow$ same

4. Random variable X has pdf $f(x) = \begin{cases} ax + bx^2 & 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$

Furthermore, $P\left(X < \frac{1}{2}\right) = \frac{3}{16}$. What is the median of X?

We need: $\int_0^1 (ax + bx^2)\,dx = \frac{a}{2} + \frac{b}{3} = 1 \quad \Rightarrow \quad 3a + 2b = 6$

$\int_0^{\frac{1}{2}} (ax + bx^2)\,dx = \frac{a}{2}\cdot\frac{1}{4} + \frac{b}{3}\cdot\frac{1}{8} = \frac{a}{8} + \frac{b}{24} = \frac{3}{16} \quad \Rightarrow \quad 3a + b = \frac{9}{2}$

$\boxed{b = \frac{3}{2}, \quad a = 1}$

Median: $\int_0^{\eta} \left(x + \frac{3}{2}x^2\right)dx = \frac{1}{2}\eta^2 + \frac{1}{2}\eta^3 = \frac{1}{2} \quad \Rightarrow \quad \eta^2 + \eta^3 = 1$

So we need $\eta^3 + \eta^2 - 1 = 0$, or $\eta \approx 0.755$

Exact: $\eta = \frac{1}{3}\left[-1 + \sqrt[3]{\frac{25}{3} - \frac{3\sqrt{69}}{2}} + \sqrt[3]{\frac{1}{2}\left(25 + 3\sqrt{69}\right)}\right]$

5. Let Y be a random variable with pdf given by $f(y) = \begin{cases} \frac{y}{2} & \text{if } 0 \le y \le 2, \\ 0 & \text{otherwise.} \end{cases}$

This is a bonus problem that wasn't on the in-class worksheet.

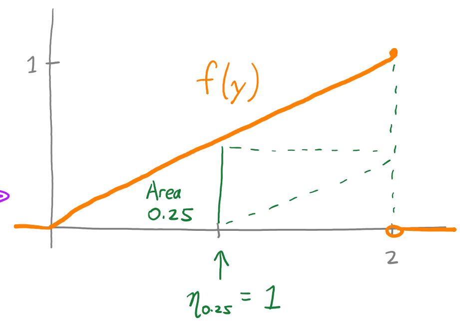
(a) Find a value $\eta_{0.25}$ such that $P(Y \le \eta_{0.25}) = 0.25$.

$\frac{1}{4} = \int_{-\infty}^{\eta_{0.25}} f(y)\,dy = \int_0^{\eta_{0.25}} \frac{y}{2}\,dy = \frac{y^2}{4}\Big|_0^{\eta_{0.25}} = \frac{\eta_{0.25}^2}{4}$

$\Downarrow$

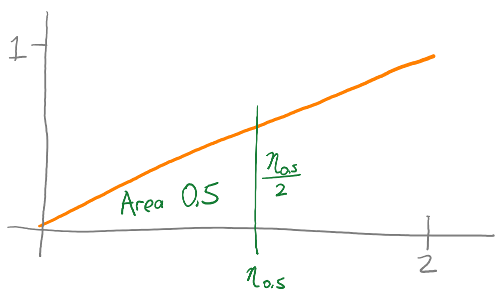
$1 = \eta_{0.25}$

or use geometry $\Rightarrow$



(b) What is the median of Y?

geometrically:



$\frac{1}{2} = \eta_{0.5}\left(\frac{\eta_{0.5}}{2}\right)\frac{1}{2}$

$2 = \eta_{0.5}^2$

$\sqrt{2} = \eta_{0.5} = \text{median}(Y)$