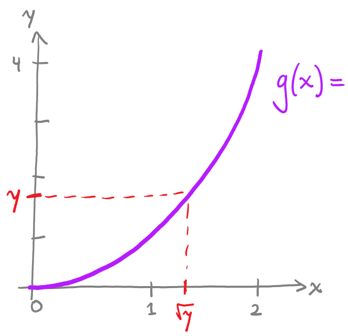
1. Let X have density $f_X(x) = \frac{x}{2}$ for $0 \le x \le 2$, and let $Y = X^2$. What is the density of Y?

Graph of Transformation



NOTE: Y takes values $0 \le y \le 4$

Find cdf of Y: for $y \in [0,4]$:

$$F_Y(y) = P(Y \le y) = P(X \le \sqrt{y})$$

$$= \int_0^{\sqrt{y}} \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_0^{\sqrt{y}} = \frac{y}{4} - 0 = \frac{y}{4}$$

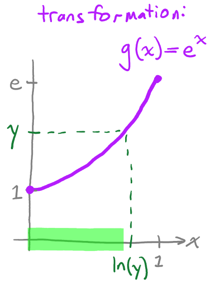
Differentiate to obtain the pdf of Y:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}\left(\frac{y}{4}\right) = \frac{1}{4} \quad \text{for } 0 \le y \le 4$$

2. Let X have density $f_X(x) = 2x$ for $0 \le x \le 1$, and let $Y = e^X$. What is the density of Y?

$g(x) = e^x$

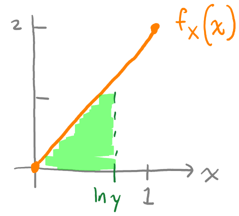
sketch the transformation:



find the cdf of Y: for $y \in [1, e]$,

$$F_Y(y) = P(Y \le y) = P(e^X \le y) = P(X \le \ln(y))$$

$$= \int_0^{\ln y} 2x\,dx = x^2\Big|_0^{\ln y} = (\ln y)^2$$



differentiate to obtain the pdf of Y:

$$f_Y(y) = \frac{d}{dy}F_Y(y) = \frac{d}{dy}(\ln y)^2 = 2\ln(y)\cdot\frac{1}{y}$$

$$f_Y(y) = \frac{2}{y}\ln(y) \quad \text{for } \underline{1 \le y \le e}$$

bounds are important!

Or use the Transformation Theorem:

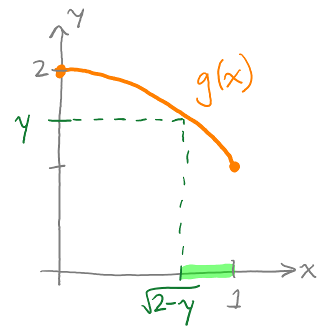
$g(x) = e^x$, which is strictly increasing on $0 \le x \le 1$

inverse is $h(y) = \ln y$, which is differentiable

thus:

$$f_Y(y) = f_X(h(y))\,|h'(y)| = 2(\ln y)\left|\frac{1}{y}\right| = \frac{2}{y}\ln y \quad \text{for } 1 \le y \le e$$

3. Let X have density $f_X(x) = 2x$ for $0 \le x \le 1$, and let $Y = 2 - X^2$. What is the density of Y?



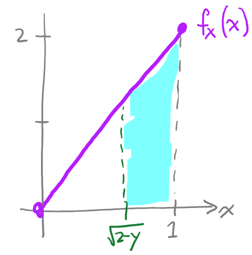
$$\left[\begin{array}{c} 0 \le Y \le y \\ \text{iff} \\ \sqrt{2-y} \le X \le 1 \end{array}\right]$$

Note that $1 \le Y \le 2$. Then for $y \in [1, 2]$:

$$F_Y(y) = P(Y \le y) = P(2 - X^2 \le y) = P(\sqrt{2-y} \le X)$$

$$= \int_{\sqrt{2-y}}^{1} f_X(x)\,dx = \int_{\sqrt{2-y}}^{1} 2x\,dx$$

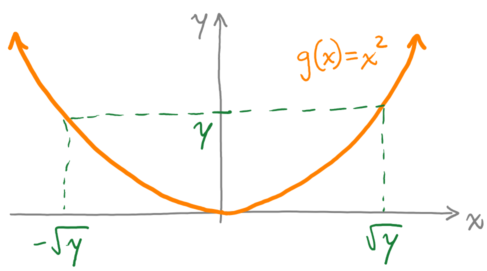
$$= x^2 \Big|_{\sqrt{2-y}}^{1} = 1 - (2-y) = y - 1$$



Then: $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(y-1) = 1$

So: $f_Y(y) = 1$ for $1 \le y \le 2$

4. Let $X \sim N(0,1)$ and $Y = X^2$. What is the distribution of Y?



Note that $Y \ge 0$.

For $y \ge 0$, $F_Y(y) = P(Y \le y) = P(X^2 \le y)$

$$= P(-\sqrt{y} \le X \le \sqrt{y}) = \int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\,dx.$$

Then $f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{1}{\sqrt{2\pi}} \left(e^{-y/2} \frac{1}{2\sqrt{y}} + e^{-y/2} \frac{1}{2\sqrt{y}} \right) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{y}{2}}$

by the Fundamental Theorem of Calculus

Thus $f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2}$ for $y \ge 0$.

This is the pdf of the Gamma$(\alpha = \frac{1}{2}, \beta = 2)$ distribution, which is also the chi-square distribution with 1 degree of freedom.