# LINEAR COMBINATIONS OF RANDOM VARIABLES

**EXPECTED VALUE:** Expected value is linear!

$$E(a_1X_1 + a_2X_2 + \cdots + a_nX_n + b) = a_1E(X_1) + a_2E(X_2) + \cdots + a_nE(X_n) + b$$

**VARIANCE:** Variance is not linear

$$Var(a_1X_1 + a_2X_2 + \cdots + a_nX_n + b) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_ia_j\, Cov(X_i, X_j)$$

$$= \sum_{i=1}^{n} a_i^2\, Var(X_i) + 2\sum_{i<j}\sum a_ia_j\, Cov(X_i, X_j)$$

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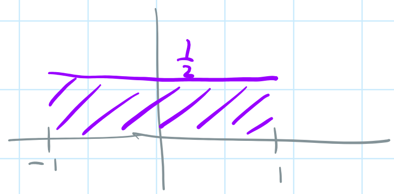
1. $X \sim Unif[-1,1] \qquad Y = X^2$




$$E(X) = 0$$

$$E(Y) = E(X^2) = \int_{-1}^{1} x^2 \cdot \tfrac{1}{2}\, dx = \tfrac{1}{3}$$

$$E(XY) = E(X^3) = \int_{-1}^{1} x^3 \cdot \tfrac{1}{2}\, dx = \frac{x^4}{8}\Big|_{-1}^{1} = \frac{1^4}{8} - \frac{(-1)^4}{8} = 0$$

So: $E(XY) = E(X)E(Y)$   Thus, $X$ and $Y$ are <u>uncorrelated</u>.

$$0 = \tfrac{1}{3} \cdot 0$$

Since $X$ determines $Y$, these variables are <u>dependent</u>.

This is an example of uncorrelated but dependent random variables.

Another example: $U \sim \mathrm{Unif}[0, 2\pi]$

Then let $X = \cos(U)$ and $Y = \sin(U)$.

This $X$ and $Y$ are uncorrelated but dependent.

**If $X$ and $Y$ are independent, then $\mathrm{Cov}(X,Y) = 0$.**

**But $\mathrm{Cov}(X,Y) = 0$ does not mean that $X, Y$ are independent.**

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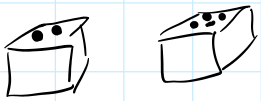
**I.**

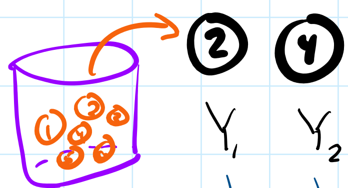


$X_1$ $X_2$

| values | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| probabilities | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{1}{6}$ |

DISCRETE UNIFORM DISTRIBUTION

**II.**



$Y_1$ $Y_2$

individually, these have the same distribution

$$E(X_i) = 1\cdot\tfrac{1}{6} + 2\cdot\tfrac{1}{6} + 3\cdot\tfrac{1}{6} + 4\cdot\tfrac{1}{6} + 5\cdot\tfrac{1}{6} + 6\cdot\tfrac{1}{6} = 3.5 = E(Y_i)$$

$= \frac{7}{2}$

$i=1$ or $i=2$

$$E(X_i^2) = 1^2\cdot\tfrac{1}{6} + 2^2\cdot\tfrac{1}{6} + 3^2\cdot\tfrac{1}{6} + 4^2\cdot\tfrac{1}{6} + 5^2\cdot\tfrac{1}{6} + 6^2\cdot\tfrac{1}{6} = \frac{91}{6}$$

$$\mathrm{Var}(X_i) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12} = \mathrm{Var}(Y_i)$$

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$$E(X_1 + X_2) = E(X_1) + E(X_2) = \frac{7}{2} + \frac{7}{2} = 7$$

$$\underbrace{Var(X_1 + X_2) = Var(X_1) + Var(X_2)}_{\text{Since } X_1 \text{ and } X_2 \text{ are independent!}} = \frac{35}{12} + \frac{35}{12} = \frac{35}{6}$$

$$E(Y_1 + Y_2) = E(Y_1) + E(Y_2) = 7$$

$$Var(Y_1 + Y_2) = Var(Y_1) + Var(Y_2) + 2\,\underbrace{Cov(Y_1, Y_2)}$$

$$= \frac{35}{12} + \frac{35}{12} + 2\left(-\frac{7}{12}\right) = \frac{14}{3}$$

$$Cov(Y_1, Y_2) = E(Y_1 \cdot Y_2) - E(Y_1)E(Y_2) = \frac{35}{3} - \frac{7}{2}\cdot\frac{7}{2}$$

| $Y_2$ \ $Y_1$ | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | ✗ | 2 | 3 | 4 | 5 | 6 |
| 2 | 2 | ✗ | 6 | 8 | 10 | 12 |
| 3 | 3 | 6 | ✗ | 12 | 15 | 18 |
| 4 | 4 | 8 | 12 | ✗ | 20 | 24 |
| 5 | 5 | 10 | 15 | 20 | ✗ | 30 |
| 6 | 6 | 12 | 18 | 24 | 30 | ✗ |

all 30 products equally likely

$$E(Y_1 \cdot Y_2) = \frac{35}{3}$$