End of Semester Practice Problems

Math 262

1. Let X and Y be iid exponential rvs with parameter λ. Let (R, Θ) be the polar coordinates of (X, Y). What is the joint density of R and Θ?

Joint density of X and Y: $f(x,y) = \lambda^2 e^{-\lambda(x+y)}$

Note that $R = \sqrt{X^2+Y^2}$, $\Theta = \operatorname{atan}\left(\frac{X}{Y}\right)$, $X = R\cos\Theta$, $Y = R\sin\Theta$

The Jacobian matrix is:
$$M = \begin{bmatrix} \frac{\partial X}{\partial r} & \frac{\partial X}{\partial \theta} \\ \frac{\partial Y}{\partial r} & \frac{\partial Y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

Then the Jacobian determinant is: $\det(M) = r\cos^2\theta + r\sin^2\theta = r$

Carlton and Devore, Section 4.6 $\Rightarrow$ By the bivariate transformation theorem, the joint density of R and θ is:
$$g(r,\theta) = \lambda^2 e^{-\lambda(r\cos\theta + r\sin\theta)}|r| = \boxed{\lambda^2 r e^{-\lambda r(\cos\theta+\sin\theta)} \quad \text{for } 0<r,\ 0<\theta<\frac{\pi}{2}}$$

2. Let $X_1, X_2, \ldots, X_{10}$ be random variables denoting bids on an item that is for sale in an auction. The item will be sold to the highest bidder. If the bids are independent and uniformly distributed between 10 and 30, what is the expected value of the sale price?

For each X_i: $f_X(x) = \frac{1}{20}$, $F_X(x) = \frac{x-10}{20}$ for $10 \le x \le 30$

$Y_{10} = \max(X_i)$ has pdf $g_{10}(y) = 10\left[F_X(y)\right]^9 f_X(y)$ $\Leftarrow$ Carlton and Devore, Section 4.9

$$= 10\left[\frac{y-10}{20}\right]^9 \cdot \frac{1}{20} = \frac{(y-10)^9}{2(20)^9} \quad \text{for } 10 \le y \le 30$$

Thus, $E(Y_{10}) = \int_{10}^{30} y \cdot \frac{(y-10)^9}{2(20)^9}\,dy = \boxed{\frac{310}{11}}$

3. Suppose $f(x)$ and $g(x)$ are probability density functions. Under what conditions on the constants α and β will the function $\alpha f(x) + \beta g(x)$ be a probability density function?

Since f and g are pdf s:
$$f(x) \ge 0,\quad g(x) \ge 0,\quad \int_{-\infty}^{\infty} f(x)\,dx = 1, \quad \text{and} \quad \int_{-\infty}^{\infty} g(x)\,dx = 1$$

If $\alpha f(x) + \beta g(x)$ is a pdf, it must be that $\boxed{\alpha \ge 0}$, $\boxed{\beta \ge 0}$, and
$$1 = \int_{-\infty}^{\infty} (\alpha f(x) + \beta g(x))\,dx = \alpha\int_{-\infty}^{\infty} f(x)\,dx + \beta\int_{-\infty}^{\infty} g(x)\,dx = \alpha + \beta,$$
so $\boxed{\alpha + \beta = 1}$.

4. Let $X \sim \text{Exp}(\lambda)$ and $0 \le s \le t$. Since X is memoryless, is it true that $(X > s+t)$ and $(X > t)$ are independent events?

Memoryless Property: $P(X > s+t \mid X > t) = P(X > s)$

Since $P(X > s) \neq P(X > s+t)$, we have

$$P(X > s+t \mid X > t) \neq P(X > s+t),$$

so the events $X > s+t$ and $X > t$ are not independent.

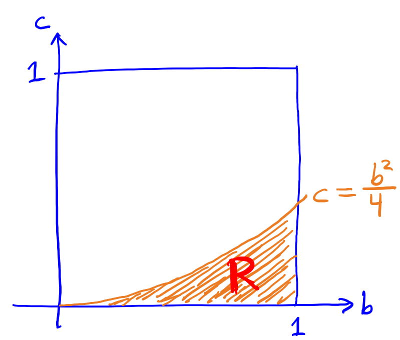
5. Suppose B and C are iid Unif$[0,1]$. Find the probability that the roots of the equation $x^2 + Bx + C = 0$ are real.

The roots are $x = \dfrac{-B \pm \sqrt{B^2 - 4C}}{2}$, which are real if and only if $B^2 - 4C \ge 0$, or equivalently, $C \le \dfrac{B^2}{4}$.

Then: $P(\text{real roots}) = P\left(C \le \frac{B^2}{4}\right)$

$$= \iint_R 1 \, dA = \text{Area}(R)$$

$$= \int_0^1 \frac{b^2}{4} \, db = \boxed{\frac{1}{12}}$$



6. Among 30 raffle tickets, six are winners. Felicia buys 10 tickets. Find the probability that she gets exactly three winners.

Let X be the number of winning tickets that Felicia buys.

Then X has a hypergeometric distribution with

$$n = 10, \quad M = 6, \quad \text{and} \quad N = 30.$$

Thus, $P(X = 3) = \dfrac{\binom{6}{3}\binom{24}{7}}{\binom{30}{10}} \approx \boxed{0.2304}$

R: dhyper(3, 6, 24, 10)