Counting Extravaganza

Math 262 • Spring 2018

1. Seven awards are to be distributed to 10 (distinguishable!) mathletes. How many different distributions are possible if:

   (a) The awards are identical and nobody gets more than one?

   Choose 7 out of 10: $\binom{10}{7} = 120$

   (b) The awards are different and nobody gets more than one?

   Permutations of 7 mathletes selected from 10: $\frac{10!}{3!} = 604800$

   (c) Awards are identical and anyone can get any number of awards?

   This is selection with replacement, when order doesn't matter, so we use the "stars and bars" method: $\binom{16}{7} = 11440$

2. At an academic conference, 6 mathematicians, 4 biologists, and 3 chemists are seated randomly in a row of 13 chairs.

   (a) How many different arrangements are possible if the chemists insist on sitting together?

   There are 11 positions for the block of 3 chairs, and 3! ways to arrange the chemists in these chairs. There are 10! ways to arrange the mathematicians and biologists in the remaining chairs. Thus, there are $11 \cdot 3!10! = 11! \cdot 3! = 239,500,800$ total arrangements.

   (b) How many different arrangements are possible if all members of the same discipline must sit together?

   There are 3! ways to arrange the three groups. With the groups, there are 6! ways to arrange the mathematicians, 4! ways to arrange the biologists, and 3! ways to arrange the chemists. Thus, there are $3!6!4!3! = 622,080$ total arrangements.

   (c) If all arrangements are equally likely, what is the probability that all members of the same discipline will sit together?

   There are 13! total arrangements, so the probability is:

$$\frac{3!6!4!3!}{13!} = 0.0000999$$

3. Consider the 20 "integer lattice points" (a, b) in the xy-plane given by $0 \leq a \leq 4$ and $0 \leq b \leq 3$, with a and b integers. (Draw a little picture.) Suppose you want to walk along the lattice points from $(0, 0)$ to $(4, 3)$, and the only legal steps are one unit to the *right* or one unit *up*.

   (a) How many legal paths are there from $(0, 0)$ to $(4, 3)$?

   Every legal path involves 7 steps, 3 of which are "up." Choose any 3 of the 7 steps to be up, and this can be done in $\binom{7}{3} = 35$ ways

   (b) How many legal paths from $(0, 0)$ to $(4, 3)$ to through the point $(2, 2)$?

   Reasoning as before, there are $\binom{4}{2} = 6$ legal paths from $(0, 0)$ to $(2, 2)$ and $\binom{3}{1} = 3$ legal paths from $(2, 2)$ to $(4, 3)$. Thus there are $6 \cdot 3 = 18$ legal paths total.

4. An urn contains 5 red, 6 white, and 7 blue balls. The urn is stirred and five balls are chosen without replacement. What is the probability that the 5 balls chosen include at least one of each color? Do this in steps:

   (a) Let E_1 be the event that *no red ball* is chosen, E_2 the event that *no white ball* is chosen, and E_3 the event that *no blue ball* is chosen. Find the probabilities $P(E_1)$, $P(E_2)$, and $P(E_3)$.

   Note that $P(E_1) = \frac{\binom{13}{5}}{\binom{18}{5}} \approx 0.150$. Similarly, $P(E_2) \approx 0.092$ and $P(E_3) \approx 0.054$.

   (b) Find the probabilities $P(E_1 \cap E_2)$, $P(E_1 \cap E_3)$, $P(E_2 \cap E_3)$, and $P(E_1 \cap E_2 \cap E_3)$.

   Note that $P(E_1 \cap E_2) = \frac{\binom{7}{5}}{\binom{18}{5}} \approx 0.002$. Similarly, $P(E_1 \cap E_3) \approx 0.00007$ and $P(E_2 \cap E_3) \approx 0.000012$. Since some balls must be chosen, $P(E_1 \cap E_2 \cap E_3) = 0$.

   (c) Use inclusion-exclusion to find $P(E_1 \cup E_2 \cup E_3)$.

   By inclusion-exclusion:

$$\begin{aligned} P(E_1 \cup E_2 \cup E_3) &= P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_1 \cap E_3) - P(E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3) \\ &\approx 0.150 + 0.092 + 0.054 - 0.002 - 0.0007 - 0.00012 \\ &\approx 0.29 \end{aligned}$$

   (d) Use the preceding result to answer the original question.

   We want $1 - P(E_1 \cup E_2 \cup E_3) \approx 0.71$.

5. An urn contains 5 red, 6 blue, and 8 green balls. If a set of 3 balls is randomly selected, what is the probability that each of the balls will be (a) of the same color; (b) of different colors? Repeat under the assumption that the balls are sampled with replacement: whenever a ball is selected, its color is noted and it is replaced in the urn before the next selection. (Hint: When sampling with replacement, each *ordered* selection is equally likely.)

   Sampling without replacement:

   (a) If the 3 balls are of the same color, then either they are all red, or all blue, or all green (three mutually exclusive options). Thus,

$$P(\text{same color}) = \frac{\binom{5}{3} + \binom{6}{3} + \binom{8}{3}}{\binom{19}{3}} = \frac{86}{969} \approx 0.089.$$

   (b) To choose three balls of different colors, we must choose 1 of 5 red balls, and 1 of 6 blue balls, and 1 of 8 green balls. Thus,

$$P(\text{different colors}) = \frac{5 \cdot 6 \cdot 8}{\binom{19}{3}} = \frac{240}{969} \approx 0.248.$$

   Sampling with replacement:

   (a) There are now 5^3 ways to choose 3 red balls, and similarly for the other colors. Thus,

$$P(\text{same color}) = \frac{5^3 + 6^3 + 8^3}{19^3} = \frac{853}{6859} \approx 0.124.$$

   (b) There are $5 \cdot 6 \cdot 8$ combinations of 3 balls, one of each color, and each combination may be ordered in $3!$ ways. Thus,

$$P(\text{different colors}) = \frac{(5 \cdot 6 \cdot 8) \cdot 3!}{19^3} = \frac{1440}{6859} \approx 0.210.$$