Linear Combinations of Random Variables

EXPECTED VALUE: linear!

$$E(a_1X_1 + a_2X_2 + \cdots + a_nX_n + b) = a_1E(X_1) + a_2E(X_2) + \cdots + a_nE(X_n) + b$$

VARIANCE: not linear!

$$Var(a_1X_1 + a_2X_2 + \cdots + a_nX_n + b) = \sum_{i=1}^{n}\sum_{j=1}^{n} \underline{a_ia_j}\, Cov(X_i, X_j)$$

$$= \sum_{i=1}^{n} \underline{a_i^2}\, Var(X_i) + 2\sum\sum_{i<j} \underline{a_ia_j}\, Cov(X_i, X_j)$$

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1. $X \sim Unif[-1, 1]$ $\quad E(X)=0$

$Y = X^2$

$$E(Y) = E(X^2) = \int_{-1}^{1} x^2 \cdot \frac{1}{2}\, dx = \frac{1}{6}x^3 \Big|_{-1}^{1} = \frac{1}{6} - \left(-\frac{1}{6}\right)$$

$$E(Y) = \frac{1}{3}$$

$$E(XY) = E(X^3) = \int_{-1}^{1} x^3 \cdot \frac{1}{2}\, dx = \frac{1}{8}x^4 \Big|_{-1}^{1} = \frac{1}{8} - \left(\frac{1}{8}\right) = 0$$

$\boxed{E(XY) = E(X)E(Y)}$ holds $\quad E(XY) = 0$

X and Y are dependent but uncorrelated!

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 0$$

and so $\text{Corr}(X, Y) = 0$

[If X and Y are independent, then $\text{Cov}(X,Y) = 0$.
But $\text{Cov}(X,Y) = 0$ does not imply independence]

Another example: $U \sim \text{Unif}[0, 2\pi]$ and

$$X = \cos(U), \quad Y = \sin(U)$$

then X and Y are dependent but uncorrelated