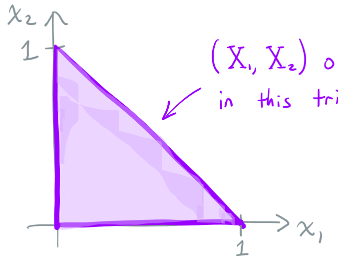
1. Let X_1 and X_2 be uniformly distributed over the region of the x_1x_2-plane defined by $0 \le x_1$, $0 \le x_2$, and $x_1 + x_2 \le 1$. Let $Y = X_1 + X_2$. Use the following steps to find the density of Y.

(a) Sketch the region of positive density for X_1 and X_2 in the x_1x_2-plane. Identify the possible values of Y.

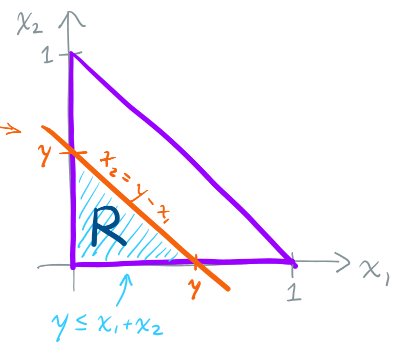


Note that X_1 and X_2 are both nonnegative and $Y = X_1 + X_2 \le 1$

Thus, $0 \le Y \le 1$.

(b) Let y be a possible value of Y. Sketch the graph $Y = y$ in the x_1x_2-plane.

Fix $y \in [0,1]$. Then $y = x_1 + x_2$, so $x_2 = y - x_1$



(c) Find the region R in the x_1x_2-plane where $Y \le y$.

R is the triangle with vertices $(0,0)$, $(y,0)$, and $(0,y)$, shaded at right.

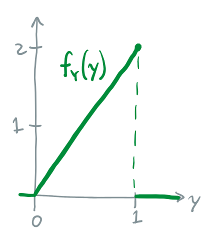
(d) Find the cdf $F_Y(y)$ by integrating the joint density of X_1 and X_2 over the region R.

$$F_Y(y) = P(Y \le X_1 + X_2) = \iint_R f(x_1, x_2)\, dA = \iint_R 2\, dA$$

$$= 2 \cdot \text{Area}(R) = 2 \cdot \frac{y^2}{2} = y^2$$

(e) Differentiate $F_Y(y)$ to obtain the density $f_Y(y)$.

$$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy}(y^2) = 2y \quad \text{for } 0 \le y \le 1$$

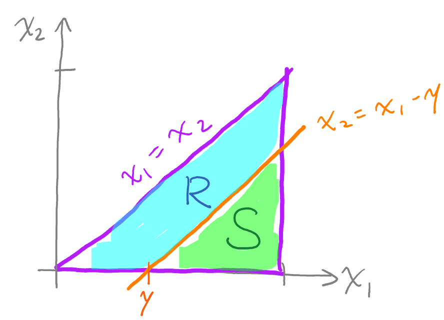


CHECK: area under $f_Y(y)$ is 1

2. Let X_1 and X_2 have joint density $f(x_1, x_2) = 3x_1$, for $0 \le x_2 \le x_1 \le 1$. Let $Y = X_1 - X_2$. Use the following steps to find the density of Y.

(a) Sketch the region of positive density for X_1 and X_2 in the x_1x_2-plane. Identify the possible values of Y.

$0 \le Y \le 1$



(b) Sketch the graph $Y = y$ in the x_1x_2-plane.

Fix $y \in [0,1]$. Then $y = x_1 - x_2$, or $\underline{\underline{x_2 = x_1 - y}}$.

(c) Find the region R in the x_1x_2-plane where $Y \le y$.

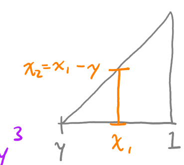
$Y = X_1 - X_2 \le y$ in the region R shaded blue.

(d) Find the cdf $F_Y(y)$ by integrating the joint density of X_1 and X_2 over the region R.

$$F_Y(y) = P(Y \le y) = P(X_1 - X_2 \le y) = \iint_R f(x_1, x_2)\, dA = 1 - \iint_S f(x_1,x_2)\, dA$$

$$= 1 - \int_y^1 \int_0^{x_1 - y} 3x_1 \, dx_2\, dx_1 = 1 - \int_y^1 3x_1(x_1 - y)\, dx_1$$

$$= 1 - \left[x_1^3 - \tfrac{3}{2}x_1^2 y\right]_{x_1=y}^{x_1=1} = 1 - \left[\left(1 - \tfrac{3}{2}y\right) - \left(y^3 - \tfrac{3}{2}y^3\right)\right] = \frac{3}{2}y - \frac{1}{2}y^3$$



One way to do this integral in Mathematica:

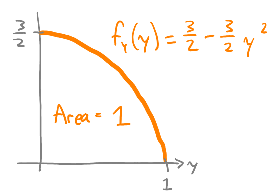
```
In[1]:= 1 - Integrate[3*x1, {x1, y, 1}, {x2, 0, x1 - y}]
```

outer bounds: {x1, y, 1}; inner bounds: {x2, 0, x1 - y}

$$\text{Out[1]=}\ \frac{3y}{2} - \frac{y^3}{2}$$

(e) Differentiate $F_Y(y)$ to obtain the density $f_Y(y)$.

$$f_Y(y) = \frac{d}{dy}\left[\frac{3}{2}y - \frac{1}{2}y^3\right] = \frac{3}{2} - \frac{3}{2}y^2 \quad \text{for } 0 \le y \le 1$$



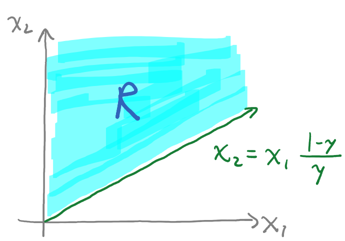
3. The joint density of X_1 and X_2 is $f(x_1, x_2) = 4e^{-2(x_1+x_2)}$. Find the density of $Y = \frac{X_1}{X_1+X_2}$.

First, note that $0 \le Y \le 1$.

For $y \in [0,1]$: $\quad Y = y \Rightarrow \dfrac{X_1}{X_1+X_2} = y \Rightarrow X_1 = X_1 y + X_2 y$

$$\Rightarrow \frac{X_1(1-y)}{y} = X_2$$

$$Y \le y \Rightarrow x_1 \frac{1-y}{y} \le x_2$$



Then: $$F_Y(y) = \iint_R f(x_1,x_2)\, dx_2 dx_1 = \int_0^\infty \int_{\frac{1-y}{y}x_1}^\infty 4e^{-2x_1 - 2x_2}\, dx_2\, dx_1 = \int_0^\infty 2e^{-2x_1/y}\, dx_1$$

inner integral:

$$\int_{\frac{1-y}{y}x_1}^\infty e^{-2x_2}\, dx_2 = -\frac{1}{2}e^{-2x_2}\Big|_{\frac{1-y}{y}x_1}^{\infty} = \frac{1}{2}e^{-2x_1\frac{1-y}{y}}$$

$$F_Y(y) = -y e^{-2x_1/y} \Big|_0^{\infty} = y \qquad \text{so } F_Y(y) = y \quad \text{for } 0 \le y \le 1.$$

Thus, $f_Y(y) = 1$ for $0 \le y \le 1$.

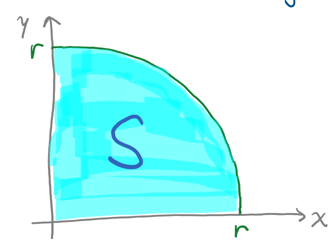
NOTE: X_1, X_2 are iid $\text{Exp}(2)$.
Y is the proportion of the sum $X_1 + X_2$ due to X_1.

4. **Challenge:** Let the point (X, Y) be randomly selected in the first quadrant of the xy-plane according to the density $f(x,y) = \frac{4}{\pi} e^{-x^2-y^2}$. Let R be the distance from (X, Y) to the origin. Find the density of R.

First, note that $0 \le R < \infty$.

Let $r > 0$. Then:

Let S be the set of points in the first quadrant at distance less than or equal to r from the origin:



$$F_R(r) = P(R \le r) = P\big((X,Y) \in S\big) = \iint_S f(x,y)\,dy\,dx$$

$$= \iint_S \frac{4}{\pi} e^{-x^2-y^2}\,dy\,dx = \underbrace{\int_0^{\frac{\pi}{2}} \int_0^r \frac{4}{\pi} e^{-t^2}\, t\,dt\,d\theta}_{\text{using polar coordinates } (t,\theta)} = 1 - e^{-r^2}$$

(radius t since r is already used)

Thus: $f_R(r) = \frac{d}{dr} F_R(r) = \frac{d}{dr}\left(1 - e^{-r^2}\right) = 2r e^{-r^2}$ for $r \ge 0$.