Homework 9

Math 330

Type (in LaTeX) your solutions to the following problems. Submit them either on Moodle or in the homework mailbox (RMS level 3, near the fireplace) by 4:00pm on **Thursday, November 16**.

The first three exercises are from the textbook:

1. Problem 6.2.6

2. Problem 6.2.7
Hint: Write the truncation error using the remainder term from Taylor's Theorem, as in Equation (6.2.3).

3. Problem 6.3.7
Hint: Modify the *Mathematica* notebook from class on Nov. 9.

4. In class, we derived the partial difference equation for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = 0, \qquad u(L,t) = 0, \qquad u(x,0) = f(x)$$

using a forward difference in time for $\frac{\partial u}{\partial t}$. This resulted in an explicit numerical scheme.

(a) This time, use a backward difference in time for $\frac{\partial u}{\partial t}$. In your answer, write the implicit scheme in the form

$$u_j^{(m-1)} = \dots$$

(b) Write the partial difference equation as a vector equation:

$$\mathbf{U}^{(m-1)} = \mathbf{A}\mathbf{U}^{(m)}$$

(c) Modify the *Mathematica* code from class (on Nov. 9) to run the implicit scheme that you devised in part (b). You may use the same initial condition as was used in for the explicit scheme. You will have to use the command `Inverse[A]` to calculate the inverse of a matrix.

(d) If you increase the step size in your code for the implicit scheme, you should see that the numerical scheme is still stable (solutions do not grow without bound). Perform a stability analysis (as in class) to show that solutions converge for all s. (This scheme is said to be *unconditionally stable.*)

5. Consider the following population dispersion model with a growth term. (This model assumes that the population has logistic growth and the spread of the population can be modeled by diffusion.)

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \lambda u(1-u), \quad 0 < x < 1, t > 0 \\
u(0,t) &= 0 \\
u(1,t) &= 0 \\
u(x,0) &= \sin(\pi x)
\end{aligned}$$

(a) Write the partial difference equation for the PDE above using a forward difference quotient for the derivative with respect to t. (This is similar to what we did this in class, but now you must include the source term).

(b) Using *Mathematica* code from class as a template, find an approximate solution to the the PDE above. For this, let $\lambda = 1$. What happens as $t \to \infty$?

(c) Now let $\lambda = 20$. What happens as $t \to \infty$? What if you use the initial condition $u(x, 0) = 0.1 \sin(\pi x)$?

(d) For the long-term behavior obtained in parts (b) and (c), explain in two sentences or less why your answers seem reasonable (relate your solutions to the physical context).