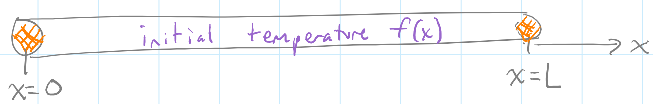
# HEAT EQUATION: $\boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}}$ (no source)

Finding a particular solution requires one initial condition and two boundary conditions.

Initial Condition: $(t=0)$ $\qquad u(x,0) = f(x)$ ← Initial temperature distribution in the rod

Boundary conditions: $\begin{pmatrix} x=0 \\ x=L \end{pmatrix}$ ONE OPTION $\left.\begin{array}{l} u(0,t) = T_1(t) \\ u(L,t) = T_2(t) \end{array}\right\}$ Specifies temperature of the endpoints at time $t$




For now, we will (mostly) ignore the initial conditions and find steady-state solutions that don't depend on time.

## EXAMPLE 1: fixed temperature at endpoints

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \boxed{u(0,t) = T_1 \quad \text{and} \quad u(L,t) = T_2}$$

Dirichlet boundary conditions

**Look for a solution that does not depend on time: $u(x,t) = u(x)$.**

Solution: since $u(x,t) = u(x)$, $\frac{\partial u}{\partial t} = 0$ and the heat eq. becomes

$$0 = k\frac{\partial^2 u}{\partial x^2} \quad \text{or} \quad 0 = \frac{\partial^2 u}{\partial x^2}$$

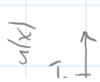
general solution: $u(x) = ax + b$

Consider the endpoints:

$u(0) = T_1 \;\Rightarrow\; T_1 = a(0) + b$, so $T_1 = b$

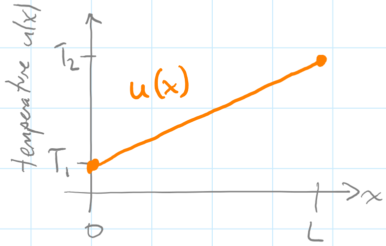
$u(L) = T_2 \;\Rightarrow\; T_2 = a(L) + b$, so $T_2 = aL + T_1$

then $\frac{T_2 - T_1}{L} = a$



then $\frac{T_2 - T_1}{L} = a$

Particular Solution: $\boxed{u(x) = \frac{T_2 - T_1}{L} x + T_1}$




## EXAMPLE 2: insulated boundaries

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \boxed{\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L,t) = 0}$$

Neumann boundary conditions

endpoints insulated: no heat flow (flux) through the ends

Fourier's law: $\underbrace{\phi(x,t)}_{\text{flux}} = -K_0 \frac{\partial u}{\partial x}(x,t)$

we have: $\phi(0,t) = 0 = -K_0 \frac{\partial u}{\partial x}(0,t)$ and also $0 = -K_0 \frac{\partial u}{\partial x}(L,t)$

$0 = \frac{\partial u}{\partial x}(0,t)$ and $0 = \frac{\partial u}{\partial x}(L,t)$

Again, look for steady-state solution $u(x,t) = u(x)$.

Solution: As before, $0 = \frac{\partial^2 u}{\partial x^2}$ has general solution $u(x) = ax + b$

$\frac{\partial u}{\partial x} = a$

Boundary conditions: $\frac{\partial u}{\partial x}(0) = 0$ so $a = 0$

$\frac{\partial u}{\partial x}(L) = 0$ so $a = 0$

So: $\boxed{u(x) = b}$

**Can we determine b?** Suppose the initial condition is $u(x,0) = f(x)$ (not necessarily constant)

and $\lim_{t\to\infty} u(x,t) = b$.

Conservation of energy:

$$\text{total thermal energy} = \int_0^L c \rho f(x)\, dx = \int_0^L c \rho b \, dx$$

mass density

specific heat

temperature at $t=0$

rod is completely insulated

temperature as $t \to \infty$

Solve for b: $\int_0^L \cancel{c}\cancel{\rho} f(x)\, dx = \int_0^L \cancel{c}\cancel{\rho}\, b \, dx$ (Assume $c, \rho$ constants)

$$\int_0^L f(x)\, dx = \int_0^L b \, dx$$

$\int_0^L f(x)\, dx = \ldots$

$$\int_0 f(x)\,dx = \quad b\,dx$$

$$\int_0^L f(x)\,dx = bL$$

Thus, $\boxed{b = \frac{1}{L}\int_0^L f(x)\,dx}$ $\leftarrow$ average value of $f$ on $[0,L]$

Particular Solution: $u(x) = \frac{1}{L}\int_0^L f(x)\,dx$

## WORKSHEET

**1.** Find steady-state solution if no sources, $\frac{\partial u}{\partial x}(0) = \alpha$, $u(L) = \beta$.

Since $\frac{\partial^2 u}{\partial x^2} = 0$, $\quad u(x) = c_1 x + c_0$

$\frac{\partial u}{\partial x}(0) = c_1 = \alpha$

$u(L) = c_1 L + c_0 = \beta$

$c_0 = \beta - c_1 L = \beta - \alpha L$

Thus, $\boxed{u(x) = \alpha x + \beta - \alpha L}$

**2.** $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 1$, $\quad u(x,0) = f(x)$, $\quad \frac{\partial u}{\partial x}(0,t) = 1$, $\quad \frac{\partial u}{\partial x}(L,t) = \beta$

and assume $K_0 = 1$

**(a)** If $\frac{\partial u}{\partial t} = 0$, then $\frac{\partial^2 u}{\partial x^2} + 1 = 0$, or $\frac{\partial^2 u}{\partial x^2} = -1$.

gen. solution: $u(x) = -\frac{1}{2}x^2 + c_1 x + c_0$

$\frac{\partial u}{\partial x} = -x + c_1$

$\frac{\partial u}{\partial x}(0,t) = c_1 = 1$

$\frac{\partial u}{\partial x}(L,t) = -L + c_1 = \beta \Rightarrow \beta = 1 - L$

**(b)** Source generates $L$ inside rod, $1$ leaves through left, $L-1$ leaves through right

**(c)** Conservation of energy: $\frac{d}{dt}\int_0^L e(x,t)\,dx = -K_0 \frac{\partial u}{\partial x}(0,t) + K_0 \frac{\partial u}{\partial t}(L,t) + \int_0^L Q\,dt = 0$

$$-1 + \beta + L = 0$$

Thus, $\beta = 1 - L$.

**(d)** Initial energy $= \int_0^L c\rho\, f(x)\,dx = \int_0^L c\rho\, u(x)\,dx =$ final energy

Thus: $\int_0^L f(x)\,dx = \int_0^L \left(-\frac{1}{2}x^2 + x + c_0\right)dx = -\frac{L^3}{6} + \frac{L^2}{2} + c_0 L$ and solve for $c_0$.