ORTHOGONALITY

PROBLEM I: recall series solution $u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

1. $f(x) = 5\sin\left(\frac{3\pi x}{L}\right)$ so $u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) = \sin\left(\frac{3\pi}{L}x\right)$

$$B_1\sin\left(\frac{\pi}{L}x\right) + B_2\sin\left(\frac{2\pi}{L}x\right) + \underbrace{B_3\sin\left(\frac{3\pi}{L}x\right)} + B_4\sin\left(\frac{4\pi}{L}x\right) + \cdots = \underbrace{5\sin\left(\frac{3\pi}{L}x\right)}$$

so $B_3 = 5$

and $B_n = 0$ for $n \neq 3$

2. $f(x) = 5\sin\left(\frac{3\pi}{L}x\right) + 8\sin\left(\frac{6\pi}{L}x\right)$ then $B_3 = 5$, $B_6 = 8$, and $B_n = 0$ for $n \notin \{3,6\}$

product-to-sum identity

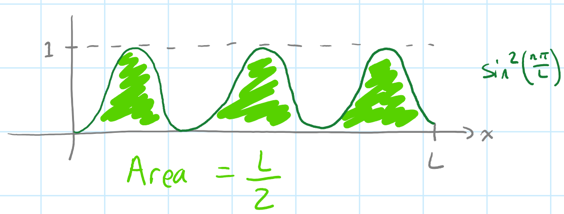
3. If $n \neq m$: $\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = -\frac{1}{2}\int_0^L\left(\cos\frac{(m+n)\pi}{L}x - \cos\frac{(n-m)\pi}{L}x\right)dx$

$$= \left[\frac{-L}{2(m+n)\pi}\sin\frac{(m+n)\pi}{L}x + \frac{L}{2(n-m)\pi}\sin\frac{(n-m)\pi}{L}x\right]_0^L$$

$$= \Bigg[\underbrace{\frac{-L}{2(m+n)\pi}\sin\big(\overset{\text{integer}}{(m+n)}\pi\big) + \frac{L}{2(n-m)\pi}\sin\big(\overset{\text{integer}}{(n-m)}\pi\big)}_{x=L} - \underbrace{0}_{x=0}\Bigg] = 0$$

If $m=n$: $\int_0^L \sin^2\left(\frac{n\pi}{L}x\right)dx = \frac{1}{2}\int_0^L\left(1-\cos\frac{2n\pi}{L}x\right)dx = \left[\frac{1}{2}x - \frac{L}{4\pi n x}\sin\frac{2n\pi}{L}x\right]_0^L$

$$= \left[\frac{L}{2} - \frac{L}{4n\pi x}\sin(2n\pi) - 0\right] = \frac{L}{2}$$



4. $f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}$ multiply by $\sin\frac{m\pi x}{L}$, $m \in \mathbb{Z}$ (fixed)

integrate x from 0 to L

$$\int_0^L f(x)\sin\frac{m\pi x}{L}dx = \int_0^L \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx$$

$$\int_0^L f(x)\sin\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} B_n \int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx$$

all terms are zero except

$\int_0^L f(x)\sin\frac{m\pi x}{L}dx = B \quad L$

$$\int_0 f(x) \sin \frac{\;}{L}\, dx = \sum_{n=1} B_n \int_0 \sin \;\; L \;\; \sin \;\; L \;\; dx$$

$$\int_0^L f(x) \sin \frac{m\pi x}{L}\, dx = B_m \frac{L}{2}$$

all terms are zero except the $n=m$ term

So $$\boxed{B_m = \frac{2}{L}\int_0^L f(x) \sin \frac{m\pi x}{L}\, dx}$$

5. $f(x) = 5$:
$L = 1$

$$B_m = 2\int_0^1 5 \sin(m\pi x)\, dx = \frac{-10}{m\pi} \cos(m\pi x) \Big|_0^1$$

$$= \frac{-10}{m\pi}\left(\cos(m\pi) - \cos(0)\right)$$

$$B_m = \frac{-10}{m\pi}\left(\cos(m\pi) - 1\right) = \begin{cases} 0 & \text{if } m \text{ is even} \\ \frac{20}{m\pi} & \text{if } m \text{ is odd} \end{cases}$$

PROBLEM II: Solution: $u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$

6. If $f(x) = 5 + \cos(2\pi x)$ and $L = 1$, then: $A_0 = 5$, $A_2 = 1$, $A_n = 0$ otherwise

7. Use $2\cos(\alpha)\cos(\beta) = \cos(\alpha - \beta) + \cos(\alpha + \beta)$

$$\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right) dx = \begin{cases} 0 & \text{if } m \neq n \\ \frac{L}{2} & \text{if } n = m \neq 0 \\ L & \text{if } n = m = 0 \end{cases}$$

8. Initial condition: $f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$

Multiply by $\cos\left(\frac{m\pi}{L}x\right)$ and integrate:

$$\int_0^L f(x) \cos\left(\frac{m\pi}{L}x\right) dx = \sum_{n=0}^{\infty} \int_0^L A_n \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right) dx$$

Then: $A_0 = \frac{1}{L}\int_0^L f(x)\, dx$ and $A_n = \frac{2}{L}\int_0^L f(x) \cos\left(\frac{n\pi}{L}x\right) dx$ if $n > 0$

9. If $L = 1$ and $f(x) = x - x^2$, then:

$$A_0 = \int_0^1 (x - x^2)\, dx = \frac{1}{6}$$

if $n > 0$: $$A_n = 2\int_0^L (x - x^2)\cos(n\pi x)\, dx = -2\left(\frac{1 + \cos(n\pi)}{n^2\pi^2}\right) = \begin{cases} 0 & \text{if } n \text{ odd} \\ \frac{-4}{n^2\pi^2} & \text{if } n \text{ even} \end{cases}$$