SUMMARY — UP TO NOW:

Heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ — separation of variables $u(x,t) = \phi(x)\,G(t)$

produces a BVP $\frac{d^2\phi}{dx^2} = -\lambda\phi$ with some conditions at $x=0$ and $x=L$.

DIRICHLET BOUNDARY CONDITIONS: $\phi(0) = \phi(L) = 0$

§2.3

eigenvalues: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, eigenfunctions: $\phi_n(x) = \sin\left(\frac{n\pi}{L}x\right)$, $n \in \{1,2,3\dots\}$

series: $f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$, where $B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$

↑ e.g. initial temperature distribution

orthogonality: $\int_0^L \sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \begin{cases} 0 & \text{if } n\neq m \\ \frac{L}{2} & \text{if } n=m \end{cases}$

NEUMANN BOUNDARY CONDITIONS: $\phi'(0) = \phi'(L) = 0$

§2.4.1

eigenvalues: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, eigenfunctions: $\phi_n(x) = \cos\left(\frac{n\pi}{L}x\right)$, $n\in\{0,1,2,\dots\}$ ($\phi_0(x) = C$)

series: $f(x) = A_0 + \sum_{n=1}^{\infty} A_n\cos\left(\frac{n\pi}{L}x\right)$, where $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$

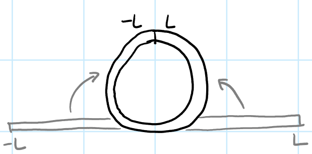
$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$ for $n\geq 1$

orthogonality: $\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = \begin{cases} 0 & \text{if } n\neq m \\ \frac{L}{2} & \text{if } n=m\neq 0 \\ L & \text{if } n=m=0 \end{cases}$

exercise 2.3.6 on HW 4

CIRCULAR BOUNDARY CONDITIONS: $\phi(-L) = \phi(L)$, $\phi'(-L) = \phi'(L)$

§2.4.2



eigenvalues: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ for $n\in\{0,1,2,\dots\}$

eigenfunctions: $\sin\left(\frac{n\pi}{L}x\right)$ and $\cos\left(\frac{n\pi}{L}x\right)$

Series: $f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi}{L}x\right)$

orthogonality: cos-cos and sin-sin integrals as above, but $-L$ to L

$\int_{-L}^{L}\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = \begin{cases} 0 & n\neq m \\ L & n=m\neq 0 \\ 2L & n=m=0\end{cases}$ $\qquad \int_{-L}^{L}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)dx = \begin{cases} 0 & n\neq m \\ L & n=m\neq 0\end{cases}$

ALSO: $\int_{-L}^{L}\sin\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)dx = 0$

Thus: $a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$, $\quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx$, $\quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$, $\quad n\geq 1$.

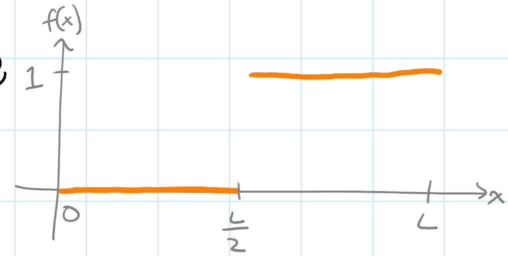
EXAMPLE: Problem II (Dirichlet boundary conditions)

with initial condition $f(x) = u(x,0) = \begin{cases} 0 & \text{if } 0 < x \le \frac{L}{2}, \\ 1 & \text{if } \frac{L}{2} < x < L. \end{cases}$

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{with} \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$$

Separate variables: $\phi(x)$ satisfies $\phi'(0) = \phi'(L) = 0$,

so I get cosine solutions:



$$\phi(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

and:
$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx = \frac{1}{L}\left(\int_0^{\frac{L}{2}} 0\,dx + \int_{\frac{L}{2}}^{L} 1\,dx\right) = \frac{1}{L}\left(0 + \frac{L}{2}\right) = \frac{1}{2}$$

for $n \ge 1$:
$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L}\int_{\frac{L}{2}}^{L} 1 \cdot \cos\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L}\cdot\frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\Bigg|_{\frac{L}{2}}^{L}$$

$$= \frac{2}{n\pi}\left(\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right) = \frac{-2}{n\pi}\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{-2}{n\pi} & \text{if } n \equiv 1 \pmod 4 \\ \frac{2}{n\pi} & \text{if } n \equiv 3 \pmod 4 \end{cases}$$

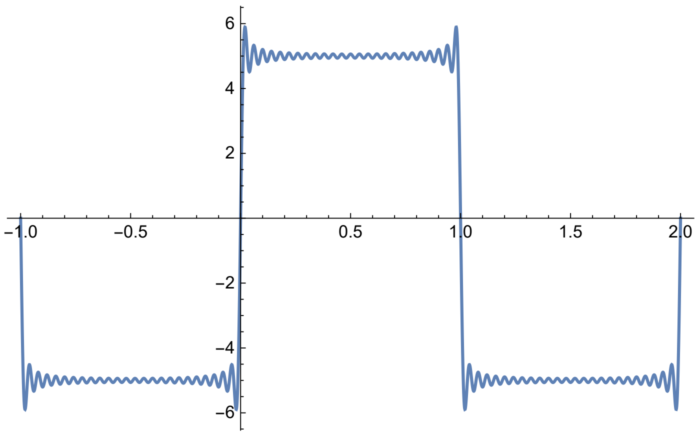
Solution:
$$u(x,t) = \frac{1}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

with A_n as defined

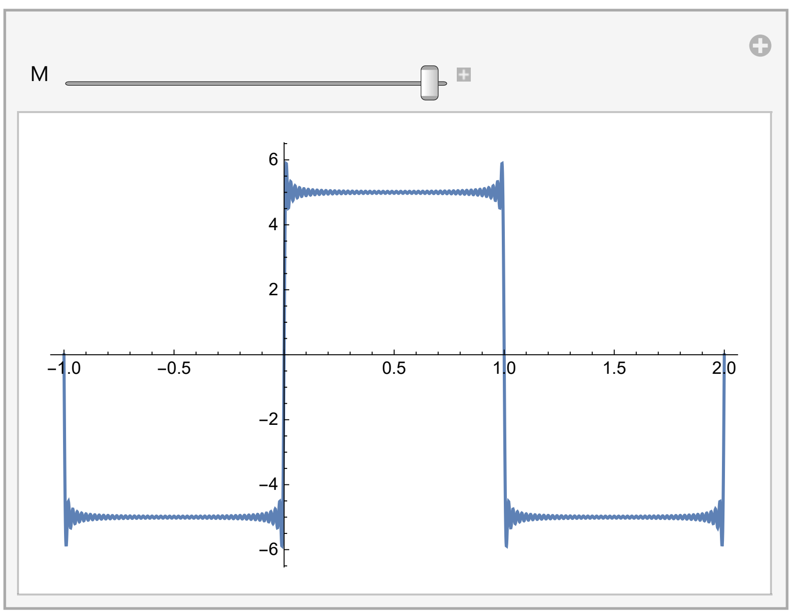
Math 330: Orthogonality

Problem I

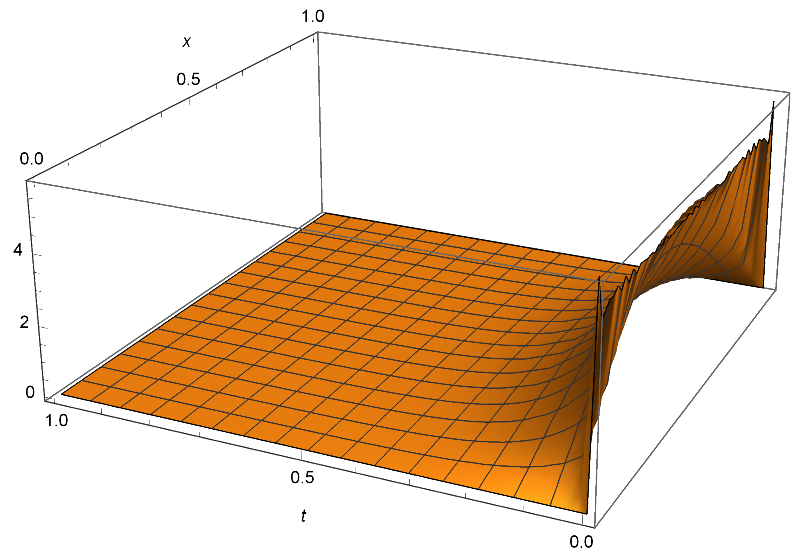
In[ ]:=
```
Plot[Sum[20/(n*Pi)*Sin[n*Pi*x], {n, 1, 50, 2}], {x, -1, 2}]
```

Out[ ]=


In[ ]:=
```
Manipulate[Plot[Sum[20/(n*Pi)*Sin[n*Pi*x], {n, 1, M, 2}], {x, -1, 2}], {M, 1, 100, 1}]
```

Out[ ]=


In[∘]:=
```
Plot3D[Sum[20/(n*Pi)*Sin[n*Pi*x]*Exp[-(n*Pi)^2*t], {n, 1, 50, 2}],
  {x, 0, 1}, {t, 0, 1}, PlotRange → All, AxesLabel → Automatic]
```

Out[∘]=


Problem II

For $n = 0$:

In[∘]:=
```
Integrate[x - x^2, {x, 0, 1}]
```

Out[∘]= $\frac{1}{6}$

For $n > 0$:

In[∘]:=
```
Integrate[2 (x - x^2) Cos[n*Pi*x], {x, 0, 1}]
```

Out[∘]= $-\frac{2\,(n\,\pi + n\,\pi\,\text{Cos}[n\,\pi] - 2\,\text{Sin}[n\,\pi])}{n^3\,\pi^3}$

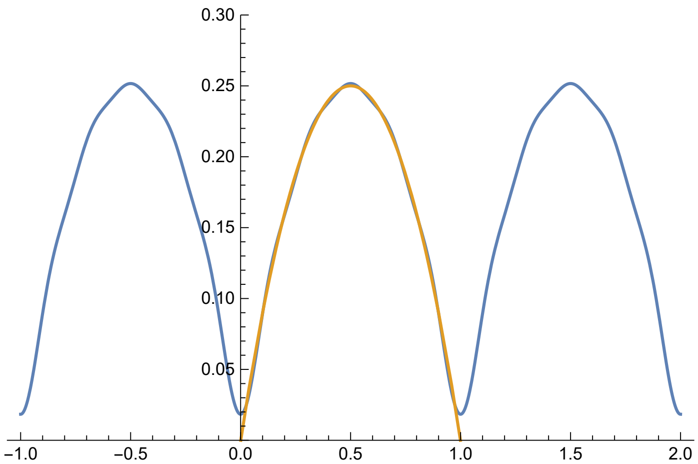
In[∘]:=
```
Simplify[Integrate[2 (x - x^2) Cos[n*Pi*x], {x, 0, 1}], Assumptions → n ∈ Integers]
```

Out[∘]= $-\frac{2\,(1 + (-1)^n)}{n^2\,\pi^2}$

In[∘]:=
```
A[n_] := -2 (1 + (-1)^n) / (n^2*Pi^2)
```

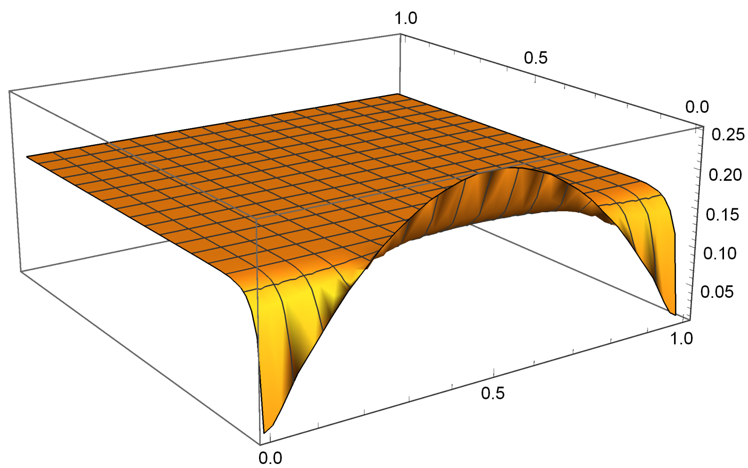
In[◦]:=
```
Plot[{1/6 + Sum[A[n] * Cos[n * Pi * x], {n, 2, 10, 2}], x - x^2},
  {x, -1, 2}, PlotRange → {0, 0.3}]
```

Out[◦]=



In[◦]:=
```
Plot3D[1/6 + Sum[A[n] Cos[n * Pi * x] Exp[-n^2 * Pi^2 * t], {n, 2, 20, 2}],
  {x, 0, 1}, {t, 0, 1}, PlotRange → All]
```

Out[◦]=



Further Examples

Problem I with piecewise-continuous initial condition

In[◦]:=
```
B[n_] = 2 * Integrate[Sin[n * Pi * x], {x, 1/2, 1}]
```

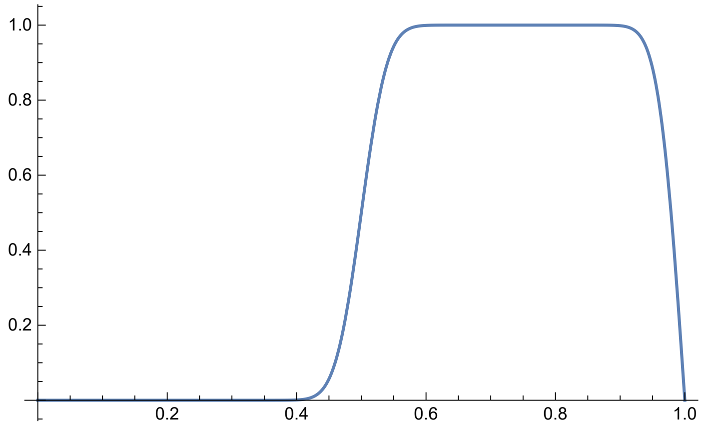
Out[◦]= $$\frac{2\left(\text{Cos}\left[\frac{n\pi}{2}\right]-\text{Cos}[n\pi]\right)}{n\pi}$$

In[◦]:=
```
Simplify[B[n], Assumptions → n ∈ Integers]
```

Out[◦]= $$\frac{2\left((-1)^{1+n}+\text{Cos}\left[\frac{n\pi}{2}\right]\right)}{n\pi}$$

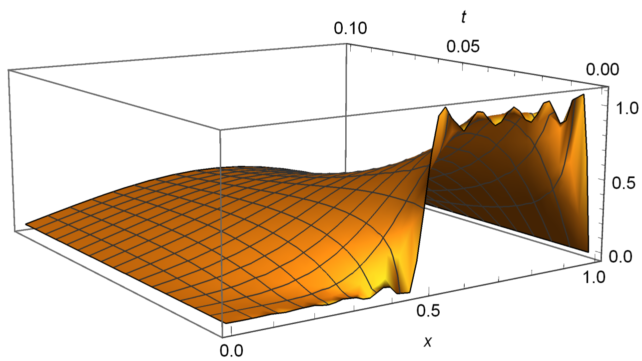
In[ ]:= `Plot[Sum[B[n] * Sin[n * Pi * x] * Exp[- (n * Pi) ^2 * 0.0005], {n, 1, 60}], {x, 0, 1}]`

Out[ ]=



In[ ]:= `Plot3D[Sum[B[n] * Sin[n * Pi * x] * Exp[- (n * Pi) ^2 * t], {n, 1, 20}], {x, 0, 1}, {t, 0, .1}, PlotRange → All, AxesLabel → Automatic]`

Out[ ]=



Problem II with piecewise continuous initial condition (let L = 1 for simplicity)

In[1]:= `A[n_] = 2 * Integrate[Cos[n * Pi * x], {x, 1 / 2, 1}]`

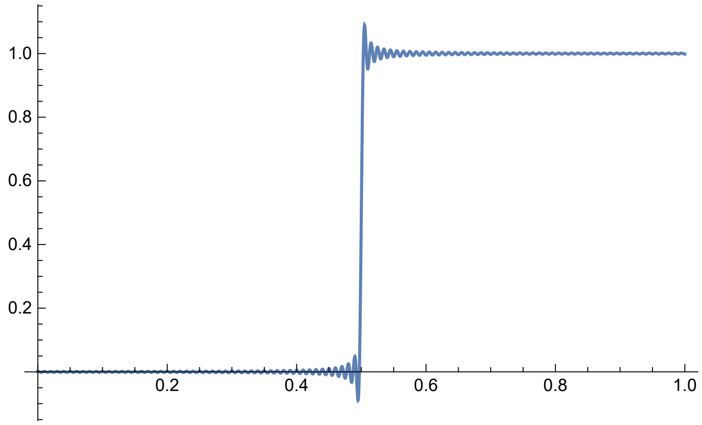
Out[1]= $\frac{2\left(-\text{Sin}\left[\frac{n\pi}{2}\right]+\text{Sin}[n\pi]\right)}{n\pi}$

In[2]:= `A[n_] = Simplify[2 * Integrate[Cos[n * Pi * x], {x, 1 / 2, 1}], Assumptions → n ∈ Integers]`

Out[2]= $-\frac{2\,\text{Sin}\left[\frac{n\pi}{2}\right]}{n\pi}$

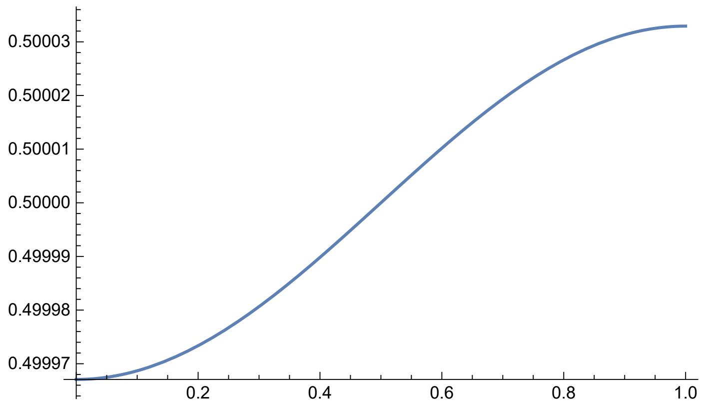
The initial condition as a sum of cosine functions. Here is a partial sum:

In[3]:=
```
Plot[1/2 + Sum[A[n] * Cos[n * Pi * x], {n, 1, 200}], {x, 0, 1}]
```

Out[3]=


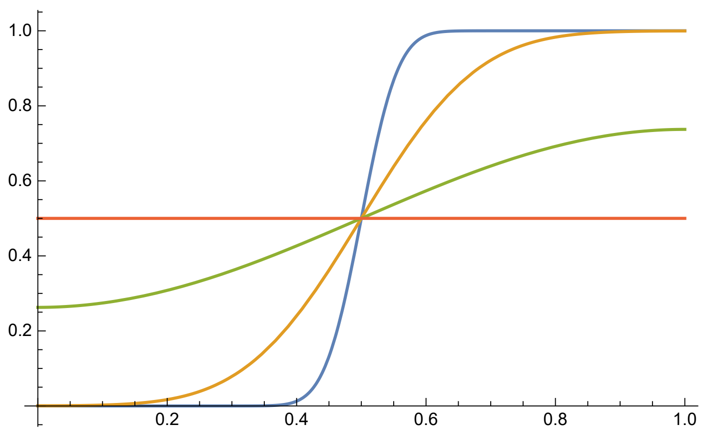
Plot the temperature function for a fixed time *t*:

In[12]:=
```
u[x_, t_] := 1/2 + Sum[A[n] * Cos[n * Pi * x] * Exp[-(n * Pi)^2 * t], {n, 1, 600}];
Plot[u[x, 1], {x, 0, 1}]
```

Out[13]=


Plot the temperature function for several fixed times:

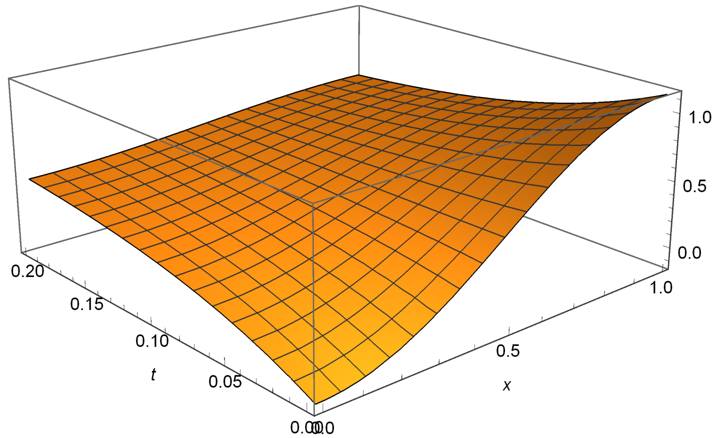
In[ ]:=
```
Plot[{u[x, 0.001], u[x, 0.01], u[x, 0.1], u[x, 1]}, {x, 0, 1}]
```

Out[ ]=


3-D plot showing temperature change over time:

In[ ]:=
```
Plot3D[1 / 2 + Sum[A[n] * Cos[n * Pi * x] * Exp[- (n * Pi) ^2 * t], {n, 1, 1}],
  {x, 0, 1}, {t, 0, 0.2}, PlotRange → All, AxesLabel → Automatic]
```

Out[ ]=



Laplace's Equation in a Rectangle — worksheet solution

1. $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\sinh x = \dfrac{e^x - e^{-x}}{2}$

(a) $\cosh 0 = 1$, $\sinh 0 = 0$

(b) $\frac{d}{dx}\cosh x = \sinh x$, $\frac{d}{dx}\sinh x = \cosh x$

(c) $\cosh^2 x - \sinh^2 x = \frac{1}{4}(e^{2x} + 2 + e^{-2x}) - \frac{1}{4}(e^{2x} - 2 + e^{-2x}) = 1$

(d) $e^x = \cosh x + \sinh x$, $e^{-x} = \cosh x - \sinh x$

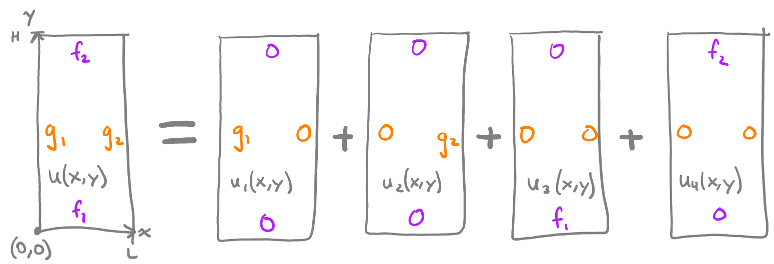
SETUP: $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0$ on $D = \{(x,y) \mid 0<x<L,\ 0<y<H\}$

$u(0,y) = g_1(y)$

$u(L,y) = g_2(y)$

$u(x,0) = f_1(x)$

$u(x,H) = f_2(x)$



2. $u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)$

If each u_i solves $\nabla^2 u$, then so does $u = \sum_{i=1}^{4} u_i$ (by superposition).

Also, $\sum_{i=1}^{4} u_i$ satisfies the boundary conditions for u.

3. Assume $u_2(x,y) = F(x)\,G(y)$. So $\nabla^2 u_2 = G(y)\frac{d^2F}{dx^2} + F(x)\frac{d^2G}{dy^2} = 0$. Then $\frac{F''}{F} = -\frac{G''}{G} = \lambda$.

Boundary:

$u_2(0,y) = F(0)\,G(y) = 0$, so $F(0) = 0$ ← Homogeneous conditions.

$u_2(L,y) = F(L)\,G(y) = g_2(y)$ ← This is the nonhomogeneous condition we will deal with later.

$u_2(x,0) = F(x)\,G(0) = 0$, so $G(0) = 0$

$u_2(x,H) = F(x)\,G(H) = 0$, so $G(H) = 0$

4. $G'' = -\lambda G$ with $G(0) = G(H) = 0$

We know that the eigenvalues are $\lambda_n = \left(\frac{n\pi}{H}\right)^2$, eigenfunctions are $\phi_n(y) = \sin\left(\frac{n\pi}{H}y\right)$, for $n \in \{1,2,3,\dots\}$

Series solution: $G(y) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{H}y\right)$

5. $F'' = \lambda F$, $F(0) = 0$

General Solution: $F(x) = c_1 e^{-\sqrt{\lambda}x} + c_2 e^{\sqrt{\lambda}x}$

or $F(x) = a_1 \cosh(\sqrt{\lambda}x) + a_2 \sinh(\sqrt{\lambda}x)$

$\hookrightarrow F(0)=0$ implies $a_1 = 0$

So $F(x) = a_2 \sinh\left(\frac{n\pi}{H} x\right)$

6. Series solution to the PDE: $u_2(x,y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{H} y\right) \sinh\left(\frac{n\pi}{H} x\right)$ ← satisfies $\nabla^2 u = 0$ and 3 homogeneous boundary conditions

7. We want $u_2(L,y) = g_2(y)$. So $g_2(y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{H} y\right) \sinh\left(\frac{n\pi}{H} L\right)$

Multiply and integrate: $\int_0^H g_2(y) \sin\left(\frac{m\pi}{H} y\right) dy = \sum_{n=1}^{\infty} \int_0^H \underbrace{A_n \sinh\left(\frac{n\pi}{H} L\right)}_{\text{constant}} \sin\left(\frac{n\pi}{H} y\right) \sin\left(\frac{m\pi}{H} y\right) dy$

By orthogonality: $\int_0^H g_2(y) \sin\left(\frac{m\pi}{H} y\right) dy = A_m \sinh\left(\frac{m\pi}{H} L\right) \cdot \frac{H}{2}$ ← orthogonality selects the m-indexed term

Therefore: $A_n = \frac{2}{H \sinh\left(\frac{n\pi L}{H}\right)} \int_0^H g_2(y) \sin\left(\frac{n\pi}{H} y\right) dy$