Math 330 — 10 October 2019

FOURIER SERIES:
$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right), \qquad -L \le x \le L$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx, \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

FOURIER COSINE SERIES:
$$f(x) \sim \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right), \qquad 0 \le x \le L$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\,dx, \qquad A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx$$

FOURIER SINE SERIES:
$$f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right), \qquad 0 \le x \le L$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

DIFFERENTIATION OF SERIES FOR e^x

Cosine series on $[0, L]$:

$$A_0 = \frac{1}{L}\int_0^L e^x\,dx = \frac{1}{L}\left(e^L - e^0\right)$$

$n \ge 1$:
$$A_n = \frac{2}{L}\int_0^L e^x \cos\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L}\left[e^x \frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)\right]_0^L - \frac{L}{n\pi}\frac{2}{L}\underbrace{\int_0^L e^x \sin\left(\frac{n\pi}{L}x\right)dx}$$

Integration by Parts:
$\int u\,dv = uv - \int v\,du$

$u = e^x$, $du = e^x dx$, $v = \frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right)$, $dv = \cos\left(\frac{n\pi}{L}x\right)dx$

$u = e^x$, $du = e^x dx$, $v = -\frac{L}{n\pi}\cos\left(\frac{n\pi}{L}x\right)$, $dv = \sin\left(\frac{n\pi}{L}x\right)dx$

$$\frac{2}{L}\int_0^L e^x \cos\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L}\left[e^x \frac{L}{n\pi}\sin\left(\frac{n\pi}{L}x\right) + \frac{L^2}{n^2\pi^2}e^x\cos\left(\frac{n\pi}{L}x\right)\right]_0^L - \frac{L^{\cancel{2}}\,2}{n^2\pi^2\cancel{L}}\underbrace{\int_0^L e^x \cos\left(\frac{n\pi}{L}x\right)dx}$$

add

$$\left(\frac{2}{L} + \frac{2L}{n^2\pi^2}\right)\int_0^L e^x\cos\left(\frac{n\pi}{L}x\right)dx = \left(0 + \frac{2}{L}\frac{L^2}{n^2\pi^2}e^L\cos(n\pi)\right) - \left(0 + \frac{2}{L}\frac{L^2}{n^2\pi^2}\right)$$

$$\int_0^L e^x\cos\left(\frac{n\pi}{L}x\right)dx = \frac{2\frac{L^2}{n^2\pi^2}e^L(-1)^n - 2\frac{L^2}{n^2\pi^2}}{\frac{2}{L} + \frac{2L}{n^2\pi^2}}\cdot\frac{(Ln^2\pi^2)}{(Ln^2\pi^2)} = \frac{L^2e^L(-1)^n - L^2}{n^2\pi^2 + L^2}$$

$$\cancel{A_n = \frac{L^2}{n^2\pi^2+L^2}\left(e^L(-1)^n - 1\right)} \qquad A_n = \frac{2L}{L^2+n^2\pi^2}\left(e^L(-1)^n - 1\right)$$

$$A_n = \frac{2}{L}\int_0^L e^x\cos\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L}\frac{L^2}{n^2\pi^2+L^2}\left(e^L(-1)^n - 1\right) = \frac{2L}{n^2\pi^2+L^2}\left(e^L(-1)^n - 1\right)$$

Sine series for e^x: $\quad e^x = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$

$$B_n = \frac{2}{L}\int_0^L e^x \sin\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L^2+n^2\pi^2}\left(n\pi - e^L n\pi(-1)^n\right) = \frac{2n\pi}{L^2+n^2\pi^2}\left(1-e^L(-1)^n\right)$$

We have:

$$e^x = \frac{e^L-1}{e^L} + \sum_{n=1}^{\infty}\frac{2L}{n^2\pi^2+L}\left(e^L(-1)^n-1\right)\cos\left(\frac{n\pi}{L}x\right)$$

differentiate term by term to obtain

$$e^x = \sum_{n=1}^{\infty}\frac{2n\pi}{L^2+n^2\pi^2}\left(1-e^L(-1)^n\right)\sin\left(\frac{n\pi}{L}x\right)$$

cannot diff. sine series term by term to obtain the cosine series

In general: Let $f(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$ and $f'(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$

Then:

$$B_n = \boxed{\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx}$$

same!

$$A_0 = \frac{1}{L}\int_0^L f'(x)\,dx = \frac{1}{L}\left(f(L)-f(0)\right)$$

$$A_n = \frac{2}{L}\int_0^L f'(x)\cos\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L}\left[f(x)\cos\left(\frac{n\pi}{L}x\right)\Big|_0^L + \frac{n\pi}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx\right]$$

$u = \cos\left(\frac{n\pi}{L}x\right)$ $\quad dv = f'(x)dx$

If the derivative of the sine series for f were equal to the cosine series for f', we would need $A_0 = 0$, or $f(L)=f(0)$.

... we would also need $A_n = \frac{n\pi}{L}B_n$, which would require that

$$f(L)\cos(n\pi) - f(0) = 0$$

$$f(L)(-1)^n = f(0)$$

this only holds if $f(L) = f(0) = 0$

SUMMARY: TERM-BY-TERM DIFFERENTIATION

Let $f(x)$ be continuous and $f'(x)$ be piecewise smooth. Then:

- The Fourier cosine series for $f(x)$ can be differentiated term by term, and the result is the Fourier sine series for $f'(x)$.

- The Fourier sine series of $f(x)$ can be differentiated term by term if and only if

$f(0)=f(L)=0$, resulting in the Fourier cosine series for $f'(x)$.

- The Fourier series of $f(x)$ can be differentiated term by term iff $f(-L)=f(L)$, resulting in the Fourier series for $f'(x)$.

NOTE: These conditions match the conditions for continuity of Fourier (sine/cosine) series!

Return to series for e^x:

We know: $B_n = \frac{-n\pi}{L} A_n$ since the cosine series can be differentiated.

Also: $\boxed{A_0 = \frac{1}{L}(e^L - 1)}$

$$A_n = \frac{n\pi}{L} B_n + \frac{2}{L}\left((-1)^n e^L - 1\right)$$

$$A_n = -\left(\frac{n\pi}{L}\right)^2 A_n + \frac{2}{L}\left((-1)^n e^L - 1\right)$$

$$A_n\left(1 + \left(\frac{n\pi}{L}\right)^2\right) = \frac{2}{L}\left((-1)^n e^L - 1\right)$$

$$\boxed{A_n = \frac{\frac{2}{L}\left((-1)^n e^L - 1\right)}{1 + \left(\frac{n\pi}{L}\right)^2} = \frac{2L\left((-1)^n e^L - 1\right)}{L^2 + n^2\pi^2}}$$

EXAMPLE: Compute the Fourier cosine series for $f(x)=x$, then use this to find the Fourier sine series for $f(x)=1$.

$$x \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{L}x\right)$$

$$A_0 = \frac{1}{L}\int_0^L x\,dx = \frac{1}{L}\frac{1}{2}x^2\Big|_0^L = \frac{L}{2} - 0$$

$$A_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi}{L}x\right)dx = \frac{2}{L}\frac{L^2}{n^2\pi^2}\left(-1+(-1)^n\right)$$

$$x \sim \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L}{n^2\pi^2}\left((-1)^n - 1\right)\cos\left(\frac{n\pi}{L}x\right)$$

DIFFERENTIATE:

$$1 \sim \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(1-(-1)^n\right)\sin\left(\frac{n\pi}{L}x\right)$$

In[2]:= `Integrate[2 / L * E^x * Sin[n * Pi * x / L], {x, 0, L}]`

Out[2]= $$\frac{2\left(n\pi - e^{L}\, n\pi \,\mathrm{Cos}[n\pi] + e^{L} L\, \mathrm{Sin}[n\pi]\right)}{L^2 + n^2\pi^2}$$

In[3]:= `Integrate[x * Cos[n * Pi * x / L], {x, 0, L}]`

Out[3]= $$\frac{L^2\left(-1 + \mathrm{Cos}[n\pi] + n\pi\,\mathrm{Sin}[n\pi]\right)}{n^2\pi^2}$$

In[5]:= `Plot[Sum[2 (1 - (-1)^n) / (n * Pi) * Sin[n * Pi * x], {n, 1, 60}], {x, -4, 4}]`

Out[5]=