Eigenfunction Expansion Problem 2

Assume $k = 1$ and $L = 1$. Here are the coefficients $A_n(t)$:

In[1]:=
```
A[n_, t_] := 2 ((-1)^n - 1) / ((n * Pi)^2 - (n * Pi)^4) * (Exp[- (n * Pi)^2 * t] - Exp[-t])
```

Here is a partial sum of the series solution for $u(x, t)$:

In[2]:=
```
M = 10;
u[x_, t_] = (3 - Exp[-t]) / 2 + Sum[A[n, t] * Cos[n * Pi * x], {n, 1, M}]
```

Out[3]=

$$\frac{1}{2}\left(3 - e^{-t}\right) - \frac{4\left(-e^{-t} + e^{-\pi^2 t}\right)\mathrm{Cos}[\pi x]}{\pi^2 - \pi^4} - \frac{4\left(-e^{-t} + e^{-9\pi^2 t}\right)\mathrm{Cos}[3\pi x]}{9\pi^2 - 81\pi^4} - \frac{4\left(-e^{-t} + e^{-25\pi^2 t}\right)\mathrm{Cos}[5\pi x]}{25\pi^2 - 625\pi^4} - \frac{4\left(-e^{-t} + e^{-49\pi^2 t}\right)\mathrm{Cos}[7\pi x]}{49\pi^2 - 2401\pi^4} - \frac{4\left(-e^{-t} + e^{-81\pi^2 t}\right)\mathrm{Cos}[9\pi x]}{81\pi^2 - 6561\pi^4}$$

Plot the solution for fixed time t:

In[16]:=
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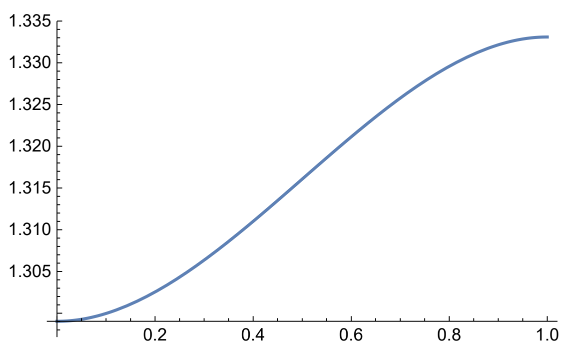
M = 10;
Plot[u[x, 1], {x, 0, 1}]
```

Out[17]=



Here is a plot of the solution for $x \in [0, 1]$ and $t \in [0, 5]$:

In[18]:=
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Plot3D[u[x, t], {x, 0, 1}, {t, 0, 5}, AxesLabel → Automatic]
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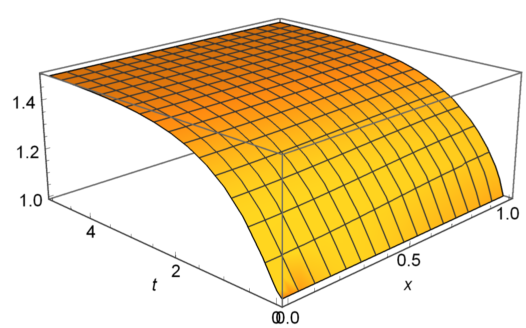
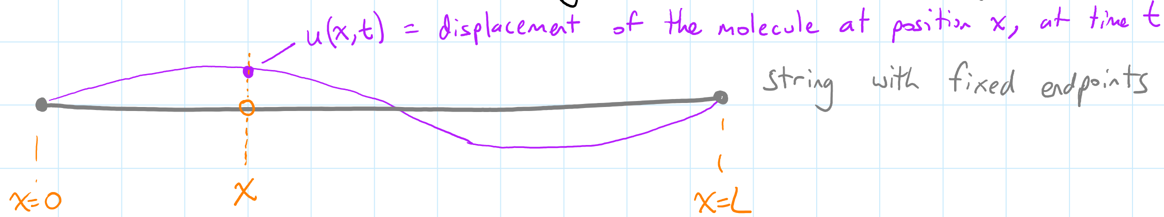
Out[18]=



WAVE EQUATION

Describes vertical vibration in a tightly-stretched string.



PDE: $$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x}}$$

WAVE EQUATION

$$c^2 = \frac{T_0}{\rho} = \frac{\text{tension}}{\text{mass density}}$$

The Wave Equation

Math 330

Note: This worksheet uses subscript notation for partial derivatives: $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, etc.

Consider the wave equation with fixed endpoints:

$$\begin{cases} u_{tt} = c^2 u_{xx} & 0 < x < L, \quad t > 0 \\ u(0,t) = 0 & t > 0 \\ u(L,t) = 0 & t > 0 \\ u(x,0) = f(x) & 0 < x < L \\ u_t(x,0) = g(x) & 0 < x < L. \end{cases} \tag{$*$}$$

1. First, we use separation of variables to solve the wave equation. Assuming that $u(x,t) = X(x)T(t)$, we arrive at the two ordinary differential equations:

$$T'' = -\lambda c^2 T \qquad \text{and} \qquad X'' = -\lambda X.$$

(a) Which of these two equations produces an eigenvalue equation? What are the eigenvalues and associated eigenfunctions?

(b) With the eigenvalues in hand, solve the other ODE. Using superposition, write down the series solution to the wave equation.

(c) Use orthogonality to determine the coefficients so that the solution satisfies the initial conditions.

2. Show that if F is any twice-differentiable function, then $u(x,t) = F(x+ct)$ and $u(x,t) = F(x-ct)$ each solve the wave equation $u_{tt} = c^2 u_{xx}$.

3. Now consider the wave equation on an *infinite string*:

$$\begin{cases} u_{tt} = c^2 u_{xx} & -\infty < x < \infty, \quad t > 0 \\ u(x,0) = f(x) & -\infty < x < \infty \\ u_t(x,0) = g(x) & -\infty < x < \infty. \end{cases}$$

(a) Consider the spacetime variables $\xi = x + ct$ and $\eta = x - ct$. Show that the PDE $u_{tt} = c^2 u_{xx}$ transforms into $u_{\xi\eta} = 0$ with these new variables.

(b) Integrate twice to show that $u_{\xi\eta} = 0$ is solved by $u(\xi, \eta) = p(\xi) + q(\eta)$.

(c) Transform your solution $p(\xi) + q(\eta)$ back to the original coordinates x and t. Can you give a physical interpretation of this solution? (*Hint*: What is the role of t?)

(d) Substitute your solution into the two initial conditions. Integrate the second expression from 0 to x. Use algebra to solve for functions p and q.

(e) Manipulate your expressions to arrive at the solution

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\, d\tau.$$

This is known as **D'Alembert's solution**.

(f) Find D'Alembert's solution using the initial condition

$$u(x,0) = \begin{cases} 1 & -1 < x < 1, \\ 0 & \text{everywhere else} \end{cases}$$
$$u_t(x,0) = 0.$$

Sketch the solution for $t = 0, 1, 2$. For concreteness, let $c = 1$.

4. Consider the wave equation which is initially unperturbed—that is, $f(x) = 0$ and everything else is as in equation $(*)$. Let $\phi(x)$ be the odd-periodic extension of $g(x)$. Show that

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \phi(\tau)\, d\tau$$

solves the wave equation with such conditions.

Hints:

(a) For all x, $\phi(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin\left(\frac{n\pi}{L} x\right)$.

(b) $\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$.

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

1. Assume $u(x,t) = X(x)\,T(t)$

Separate variables: $T'' = -\lambda c^2 T$ and $X'' = -\lambda X$

(a) Boundary conditions: $\underbrace{u(0,t) = u(L,t) = 0}$

$$\downarrow$$

$$X(0)T(t) = X(L)T(t) = 0$$

so $X(0) = X(L) = 0$

EIGENVALUE PROBLEM

$X'' = -\lambda X$

$X(0) = X(L) = 0$

eigenvalues: $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1,2,3,\ldots$

eigenfunctions: $X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$

(b) Solve: $T'' = -\lambda c^2 T$

Since we know that $\lambda > 0$ from part (a), $T_n(t) = a_n \cos(c\sqrt{\lambda}\,t) + b_n \sin(c\sqrt{\lambda}\,t)$

$$T_n(t) = a_n \cos\left(\tfrac{cn\pi}{L}t\right) + b_n \sin\left(\tfrac{cn\pi}{L}t\right)$$

Product solution: $X_n(x)\,T_n(t) = \sin\left(\frac{n\pi}{L}x\right)\left(a_n \cos\left(\frac{cn\pi}{L}t\right) + b_n \sin\left(\frac{cn\pi}{L}t\right)\right)$

Series solution: $\boxed{u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(a_n \cos\left(\frac{cn\pi}{L}t\right) + b_n \sin\left(\frac{cn\pi}{L}t\right)\right)}$

(c) Initial conditions: $\underbrace{u(x,0) = f(x)}_{\text{initial position}}$, $\underbrace{\frac{\partial u}{\partial t}(x,0) = g(x)}_{\text{initial velocity}}$

$t=0$: $f(x) = u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right)$ so $\boxed{a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx}$

differentiate: $\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left(-a_n \frac{cn\pi}{L}\sin\left(\frac{cn\pi}{L}t\right) + b_n \frac{cn\pi}{L}\cos\left(\frac{cn\pi}{L}t\right)\right)$

$g(x) = \frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} b_n \frac{cn\pi}{L}\sin\left(\frac{n\pi}{L}x\right)$ so $b_n \frac{cn\pi}{L} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$

then $\boxed{b_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx}$

2. Let $F: \mathbb{R} \to \mathbb{R}$ be a twice-differentiable function and $u(x,t) = F(x+ct)$.

Then: $u_x = F'(x+ct)$ $\qquad u_t = c\,F'(x+ct)$

$u_{xx} = F''(x+ct)$ $\qquad u_{tt} = c^2 F''(x+ct)$

Thus, $u_{tt} = c^2 u_{xx}$.

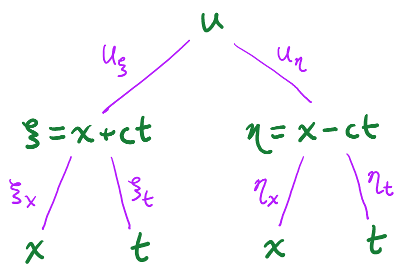
If $u(x,t) = F(x-ct)$, then $u_{xx} = F''(x-ct)$ and $u_{tt} = (-1)^2 c^2 F(x-ct)$,

so $u_{tt} = c^2 u_{xx}$.

Interpretation: F represents a traveling wave

3. (a) Define $\underbrace{\xi = x+ct, \quad \eta = x-ct}_{\text{"characteristics"}}$

Use the multivariable chain rule:



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)$$

$$= \left(\frac{\partial^2 u}{\partial \xi^2}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta \partial \xi}\cdot\frac{\partial \xi}{\partial x}\right) + \left(\frac{\partial^2 u}{\partial \xi \partial \eta}\cdot\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\cdot\frac{\partial \eta}{\partial x}\right)$$

$$= \frac{\partial^2 u}{\partial \xi^2}\cdot 1 + \frac{\partial^2 u}{\partial \eta \partial \xi}\cdot 1 + \frac{\partial^2 u}{\partial \xi \partial \eta}\cdot 1 + \frac{\partial^2 u}{\partial \eta^2}\cdot 1$$

$$= \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2}$$

Similarly, $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial t} = c\frac{\partial u}{\partial \xi} - c\frac{\partial u}{\partial \eta}$

$$\frac{\partial^2 u}{\partial t^2} = c\left(\frac{\partial^2 u}{\partial \eta^2}\cdot c + \frac{\partial^2 u}{\partial \eta \partial \xi}\cdot(-c)\right) - c\left(\frac{\partial^2 u}{\partial \xi \partial \eta}\cdot c + \frac{\partial^2 u}{\partial \eta^2}\cdot(-c)\right) = c^2\frac{\partial^2 u}{\partial \xi^2} - 2c^2\frac{\partial^2 u}{\partial \eta \partial \xi} + c^2\frac{\partial^2 u}{\partial \eta^2}$$

The wave equation becomes:
$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

$$c^2\cancel{\frac{\partial^2 u}{\partial \xi^2}} - 2c^2\frac{\partial^2 u}{\partial \eta \partial \xi} + c^2\cancel{\frac{\partial^2 u}{\partial \eta^2}} = c^2\left(\cancel{\frac{\partial^2 u}{\partial \xi^2}} + 2\frac{\partial^2 u}{\partial \eta \partial \xi} + \cancel{\frac{\partial^2 u}{\partial \eta^2}}\right)$$

$$0 = 4c^2\frac{\partial^2 u}{\partial \eta \partial \xi} \longrightarrow \boxed{0 = \frac{\partial^2 u}{\partial \eta \partial \xi}}$$ Wave equation in ξ and η.

(b) Integrate with respect to η: $\int \frac{\partial^2 u}{\partial \eta \partial \xi}\, d\eta = \int 0 \, d\eta$

$$\frac{\partial u}{\partial \xi} = r(\xi)$$ $\leftarrow$ some function of ξ

Integrate with respect to ξ: $\int \frac{\partial u}{\partial \xi}\, d\xi = \int r(\xi)\, d\xi$

$$u = p(\xi) + q(\eta)$$

antiderivative of $r(\xi)$, $p' = r$ — some function of η

(c) We have: $\boxed{u(x,t) = p(x+ct) + q(x-ct)}$ $\leftarrow$ agrees with #2

$\frac{\partial u}{\partial t} = cp'(x+ct) - cq'(x-ct)$

(d) Initial conditions: $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$

So: $u(x,0) = p(x) + q(x) = f(x)$ and $\frac{\partial u}{\partial t}(x,0) = \underbrace{cp'(x) - cq'(x) = g(x)}_{\text{integrate with respect to } x}$

$$cp(x) - cq(x) = G(x)$$

where $G'(x) = g(x)$

We have: $\begin{cases} p(x) + q(x) = f(x) \\ p(x) - q(x) = \frac{1}{c} G(x) \end{cases}$

Add to obtain: $2p(x) = f(x) + \frac{1}{c} G(x)$ so $p(x) = \frac{1}{2} f(x) + \frac{1}{2c} G(x)$

Subtract to obtain: $2q(x) = f(x) - \frac{1}{c} G(x)$ so $q(x) = \frac{1}{2} f(x) - \frac{1}{2c} G(x)$

(e) to be continued...