Wave equation.

infinite tightly stretched string

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

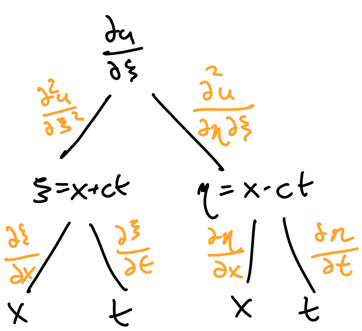
describes vertical vibration in this string

$c =$ wave speed

solution $u(t,x)$ is the vertical displacement at position x, time t

$$\boxed{\text{Wave Eq: } \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right)$$



$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right)$$

$$= \left(\frac{\partial^2 u}{\partial \xi^2}\cdot \underbrace{\frac{\partial \xi}{\partial x}}_{1} + \frac{\partial^2 u}{\partial \eta \partial \xi}\cdot \underbrace{\frac{\partial \eta}{\partial x}}_{1}\right) + \left(\frac{\partial^2 u}{\partial \xi \partial \eta}\cdot \underbrace{\frac{\partial \xi}{\partial x}}_{1} + \frac{\partial^2 u}{\partial \eta^2}\cdot \underbrace{\frac{\partial \eta}{\partial x}}_{1}\right)$$

$$\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \eta \partial \xi} + \frac{\partial^2 u}{\partial \eta^2}}$$

Similarly:

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\cdot \underbrace{\frac{\partial \xi}{\partial t}}_{c} + \frac{\partial u}{\partial \eta}\cdot \underbrace{\frac{\partial \eta}{\partial t}}_{-c} = c\frac{\partial u}{\partial \xi} - c\frac{\partial u}{\partial \eta}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t}\left(c\frac{\partial u}{\partial \xi} - c\frac{\partial u}{\partial \eta}\right) = c\left(\frac{\partial^2 u}{\partial \xi^2}\cdot \underbrace{\frac{\partial \xi}{\partial t}}_{c} + \frac{\partial^2 u}{\partial \xi \partial \eta}\cdot \underbrace{\frac{\partial \eta}{\partial t}}_{-c}\right) - c\left(\frac{\partial^2 u}{\partial \xi \partial \eta}\cdot \underbrace{\frac{\partial \xi}{\partial t}}_{c} + \frac{\partial^2 u}{\partial \eta^2}\cdot \underbrace{\frac{\partial \eta}{\partial t}}_{-c}\right)$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial \xi^2} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c^2\frac{\partial^2 u}{\partial \eta^2}}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$\cancel{c^2 \frac{\partial^2 u}{\partial \xi^2}} - 2c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \cancel{c^2 \frac{\partial^2 u}{\partial \eta^2}} = c^2 \left(\cancel{\frac{\partial^2 u}{\partial \xi^2}} + 2 \frac{\partial^2 u}{\partial \eta \partial \xi} + \cancel{\frac{\partial^2 u}{\partial \eta^2}} \right)$$

$$-2c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} = 2c^2 \frac{\partial^2 u}{\partial \eta \partial \xi}$$

$$0 = 4c^2 \frac{\partial^2 u}{\partial \eta \partial \xi}$$

$$\boxed{0 = \frac{\partial^2 u}{\partial \eta \partial \xi}} \quad \text{since} \quad c > 0$$

This is the wave equation in the characteristic variables ξ, η.

Wave Equation: d'Alembert's Solution

Math 330

1. Show that if F and G are any C^2 functions, then $u(t,x) = F(x+ct) + G(x-ct)$ ~~each~~ solves the wave equation $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$.

$$\frac{\partial u}{\partial t} = F'(x+ct)\,c + G'(x-ct)(-c)$$

$$\frac{\partial^2 u}{\partial t^2} = F''(x+ct)\,c^2 + G''(x-ct)\,c^2$$

$$\frac{\partial u}{\partial x} = F'(x+ct) + G'(x-ct)$$

$$\frac{\partial^2 u}{\partial x^2} = F''(x+ct) + G''(x-ct)$$

see that

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

2. Consider the wave equation on an *infinite string*,

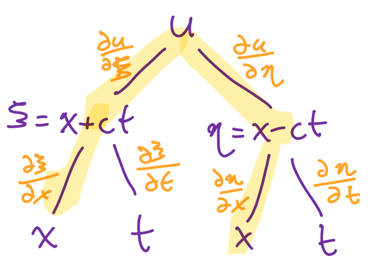
$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad -\infty < x < \infty, t \geq 0}$$

with initial conditions $\boxed{u(0,x) = f(x)}$ and $\boxed{\frac{\partial u}{\partial t}(0,x) = g(x).}$

init. position   init. velocity $t=0$

(a) Consider the spacetime variables $\xi = x + ct$ and $\eta = x - ct$. Show that the PDE $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ transforms into $\dfrac{\partial^2 u}{\partial \eta \partial \xi} = 0$ with these new variables.

xi   eta



$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\cdot\underbrace{\frac{\partial \xi}{\partial x}}_{1} + \frac{\partial u}{\partial \eta}\cdot\underbrace{\frac{\partial \eta}{\partial x}}_{1}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

Multiply Jacobian matrices:

$$\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial \xi} & \frac{\partial u}{\partial \eta} \end{bmatrix} \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial t} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial t} \end{bmatrix}$$

$$= \Big[\underbrace{\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial x}}_{\frac{\partial u}{\partial x}},\ \underbrace{\frac{\partial u}{\partial \xi}\cdot\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial t}}_{\frac{\partial u}{\partial t}}\Big]$$

(b) Integrate twice to show that $\dfrac{\partial^2 u}{\partial \eta \partial \xi} = 0$ is solved by $u(\xi, \eta) = p(\xi) + q(\eta)$.

integrate w.r.t. η: $\displaystyle\int \frac{\partial^2 u}{\partial \eta \partial \xi}\, d\eta = \int 0 \, d\eta$

$$\frac{\partial u}{\partial \xi} = r(\xi) \leftarrow \text{some function of } \xi$$

integrate w.r.t. ξ: $\displaystyle\int \frac{\partial u}{\partial \xi}\, d\xi = \int r(\xi)\, d\xi$

$$u = p(\xi) + q(\eta)$$

p is antiderivative of r $p' = r$

(c) Transform your solution $p(\xi) + q(\eta)$ back to the original coordinates x and t. Can you give a physical interpretation of this solution?

recall $\xi = x + ct$, $\eta = x - ct$

so: $u = p(\xi) + q(\eta)$

$$\boxed{u(t,x) = p(x+ct) + q(x-ct)}$$

$\frac{\partial u}{\partial t} = p'(x+ct) \cdot c + q'(x-ct)(-c)$ $\frac{\partial u}{\partial t}(0,x) = c \cdot p'(x) - c\, q'(x)$

(d) Substitute your solution into the two initial conditions. Integrate the second expression from 0 to x. Use algebra to solve for functions p and q.

initial conditions: $u(0,x) = f(x) = p(x) + q(x)$

$\frac{\partial u}{\partial t}(0,x) = g(x) = c \cdot p'(x) - c \cdot q'(x)$

We will continue this on Thursday.

(e) Manipulate your expressions to arrive at the solution

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)\, d\tau.$$

This is known as *D'Alembert's solution.*

3. Consider the following wave equation on an infinite medium ($-\infty < x < \infty$):

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \le x < 2 \\ 0, & \text{otherwise} \end{cases}, \qquad \frac{\partial u}{\partial t}(x,0) = 0$$

(a) Sketch what you think the solution should look like at $t = 0, t = 1$, and $t = 2$ (without actually finding the solution yet).

(b) Find d'Alembert's solution to this wave equation.

(c) Use software to plot d'Alembert's solution at $t = 0$, $t = 1$, and $t = 2$ to verify your earlier plot.

4. Find d'Alembert's solution to the following wave equation on an infinite medium:

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = \cos(x)$$

Use software to plot the wave profile over time.

5. For each of the following wave equations on an infinite medium, sketch (by hand) the wave profiles at $t = 0$, $t = 1$, and $t = 2$ without solving the equations. Check your sketch using software.

(a) $$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = \begin{cases} -1, & -1 < x < 0 \\ 1, & 0 \le x < 1 \\ 0, & \text{otherwise} \end{cases}, \qquad \frac{\partial u}{\partial t}(x,0) = 0$$

(b) $$\frac{\partial^2 u}{\partial t^2} = \frac{1}{4}\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = \begin{cases} x, & -1 < x < 1 \\ 0, & \text{otherwise} \end{cases}, \qquad \frac{\partial u}{\partial t}(x,0) = 0$$