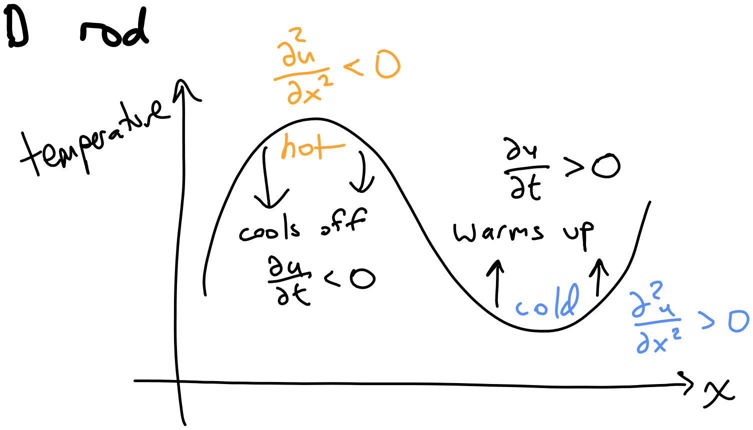
# Heat Equation $\qquad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$u(t,x)$ is temperature at position $x$, time $t$ in a 1-D rod




Suppose $u(t,x) = e^{\lambda t} v(x)$ for some function $v(x)$.

Then $\frac{\partial u}{\partial t} = \lambda e^{\lambda t} v(x)$ and $\frac{\partial^2 u}{\partial x^2} = e^{\lambda t} v''(x)$.

Heat equation: $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$

$$\lambda \cancel{e^{\lambda t}} v(x) = \cancel{e^{\lambda t}} v''(x)$$

$$\boxed{\lambda v = v''} \quad \text{ODE}$$

What are the solutions?

For what $\lambda$ is there a solution?

# Eigenvalues and Eigenfunctions

Math 330

Consider the ODE:

$$\boxed{\frac{d^2v}{dx^2} = \lambda v}$$

eigenvalue problem

Characteristic equation:

$r^2 - \lambda = 0$

**1.** What is the solution to the ODE...

(a) ... if $\lambda > 0$?

Let $\omega = \sqrt{\lambda}$.

$r = \pm \omega$

$v(x) = c_1 e^{-\omega x} + c_2 e^{\omega x}$

(b) ... if $\lambda = 0$?

$r = 0$

$v(x) = c_1 x + c_2$

(c) ... if $\lambda < 0$?

Let $\omega = \sqrt{-\lambda}$

Here, $r = \pm i\omega$.

$v'' = \lambda v$

$v(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$

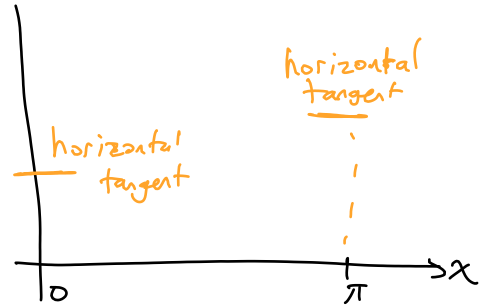
**2.** Which of the solutions you found in #1 satisfy the boundary conditions $v'(0) = 0$ and $v'(\pi) = 0$?

$\lambda > 0$ (a) No eigenvalues, eigenfunctions satisfy BC's.

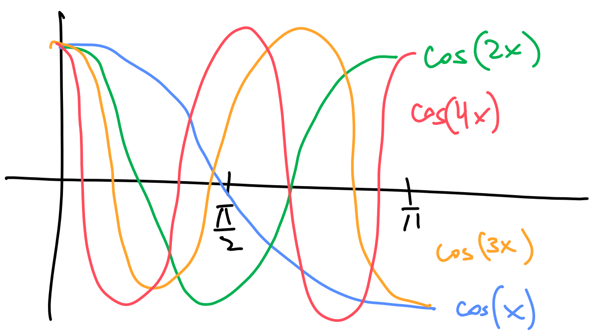
$\lambda = 0$ (b) $v(x) = c_1 x + c_2$ is solution iff $c_1 = 0$

$\boxed{v(x) = c_2}$

$\lambda < 0$ (c) $v(x) = \underbrace{c_1 \cos(\omega x)} + \underbrace{c_2 \sin(\omega x)}_{c_2 = 0}$




Need: $\cos(\omega x)$ to have horizontal tangent at $x = \pi$




Need $\omega = 1, 2, 3, \dots$

$\omega \in \mathbb{Z}^+$ pos. integer

eigenvalues: $\lambda = -\omega^2$ squares of pos. integers

eigenfunctions: $v(x) = c_1 \cos(\omega x)$

**3.** Which of the solutions you found in #1 satisfy the periodic boundary conditions $v(-\pi) = v(\pi)$ and $v'(-\pi) = v'(\pi)$?

(a) No solutions if $\lambda > 0$.

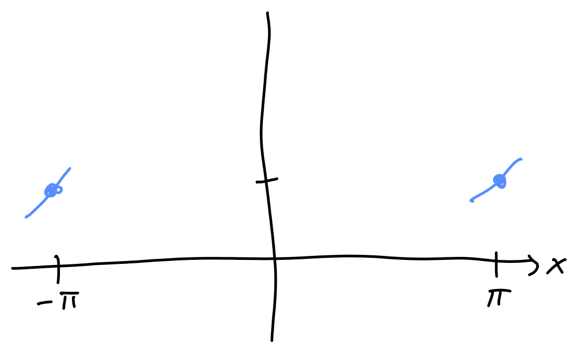
(b) If $\lambda = 0$, then $v(x) = C$ is the only solution.

(c) If $\lambda < 0$, then

$v(x) = c_1 \cos(\omega x) + c_2 \sin(\omega x)$

is a solution for any positive integer $\omega$.

In this case, $\lambda = -\omega^2$.




**Eigen solutions:** $v_k(x) = \cos(kx)$

$\tilde{v}_k(x) = \sin(kx)$

for $k \in \{0, 1, 2, 3, \dots\}$

# Orthogonality

Math 330

**1.** Let $m$ and $n$ be integers. Prove the identity

Here, m and n are integers!

$$\int_{=\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 0, \text{if } m \neq n, \\ \pi, \text{if } m = n \neq 0. \end{cases}$$

$-\pi$

RECALL: $\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

Let $\alpha = mx$, $\beta = nx$, then:

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos((m-n)x) - \cos((m+n)x)\right]dx$$

if $m \neq n$ $\longrightarrow$ $= \frac{1}{2}\left[\frac{1}{m-n}\sin((m-n)x) - \frac{1}{m+n}\sin((m+n)x)\right]_{-\pi}^{\pi} = 0$

integers

If $m = n \neq 0$: $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \int_{\pi}^{\pi} \sin^2(nx)\,dx = \int_{-\pi}^{\pi} \frac{1-\cos(2nx)}{2}\,dx = \pi$

We will continue this next time..

**2.** We would like to be able to write "any" function $f(x)$ as a sum of sine functions. Assume that $f(x)$ can be written as

$$f(x) = \sum_{k=1}^{\infty} b_k \sin(kx).$$

Multiply both sides of the equation above by $\sin(mx)$, then integrate both sides from $-\pi$ to $\pi$ with respect to $x$. Interchange integration and summation and solve for $b_k$.

**3.** Use your newfound power to write $f(x) = x$ as a sum of sine functions. Write out the first 6 terms in the series. Use technology to plot your series.

**4.** Derive an identity for cosine of the following form:

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\ dx = \begin{cases} & n \neq m \\ & n = m \neq 0 \\ & n = m = 0 \end{cases}$$

**5.** Can you write $f(x) = x$ as a sum of cosine functions? Why or why not?