Orthogonality

Math 330

1. Let m and n be integers. Prove the identity

Here, m and n are integers!

$$\langle \sin(mx), \sin(nx) \rangle = \int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \begin{cases} 0, \text{if } m \neq n, \text{ or } m=n=0, \\ \pi, \text{if } m = n \neq 0. \end{cases}$$

RECALL: $\sin(\alpha)\sin(\beta) = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

Let $\alpha = mx$, $\beta = nx$, then:

$$\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos((m-n)x) - \cos((m+n)x)\right]dx$$

if $m \neq n \rightarrow$ $$= \frac{1}{2}\left[\frac{1}{m-n}\sin((m-n)x) - \frac{1}{m+n}\sin((m+n)x)\right]_{-\pi}^{\pi} = 0$$

integers

If $m=n\neq 0$: $\int_{-\pi}^{\pi} \sin(mx)\sin(nx)\,dx = \int_{-\pi}^{\pi} \sin^2(nx)\,dx = \int_{-\pi}^{\pi} \frac{1-\cos(2nx)}{2}\,dx = \pi$

We will continue this next time..

2. We would like to be able to write "any" function $f(x)$ as a sum of sine functions. Assume that $f(x)$ can be written as

$$\longrightarrow \quad f(x) = \sum_{k=1}^{\infty} b_k \sin(kx).$$

Multiply both sides of the equation above by $\sin(mx)$, then integrate both sides from $-\pi$ to π with respect to x. Interchange integration and summation and solve for b_k.

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \int_{-\pi}^{\pi} \sum_{k=1}^{\infty} b_k \sin(kx)\sin(mx)\,dx$$

$$\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = \sum_{k=1}^{\infty} b_k \underbrace{\int_{-\pi}^{\pi} \sin(kx)\sin(mx)\,dx}_{=0 \text{ unless } m=k, \text{ in which case integral is } \pi} = b_m \cdot \pi$$

$k=m$ term of the sum

Thus: $$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,dx = b_m$$

Alternatively: $$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(kx)\,dx$$

$$x = \sum_{k=1}^{\infty} b_k \sin(kx)$$

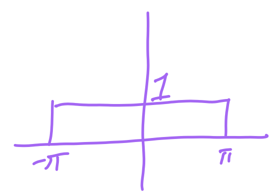
3. Use your newfound power to write $f(x) = x$ as a sum of sine functions. Write out the first 6 terms in the series. Use technology to plot your series.

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cdot \sin(kx)\, dx = \frac{1}{\pi}\left[\frac{-x}{k}\cos(kx) + \frac{1}{k^2}\sin(kx)\right]_{-\pi}^{\pi}$$

$$= \frac{-2}{k}\underbrace{\cos(k\pi)}_{\substack{1 \text{ if } k \text{ even} \\ -1 \text{ if } k \text{ odd}}} = \frac{-2}{k}(-1)^k = \frac{2}{k}(-1)^{k+1}$$

4. Derive an identity for cosine of the following form: $m, n \in \mathbb{Z}$

$$\int_{-\pi}^{\pi} \cos(mx)\cos(nx)\, dx = \begin{cases} 0 & \text{if } n \neq m \\ \pi & \text{if } n = m \neq 0 \\ 2\pi & \text{if } n = m = 0 \end{cases}$$



ALSO,
$$\int_{-\pi}^{\pi} \cos(mx)\sin(nx)\, dx = 0$$
for all integers n, m

5. Can you write $f(x) = x$ as a sum of cosine functions? Why or why not?

p. 74 in the text

FOURIER SERIES

The Fourier series of a function $f(x)$ defined on $-\pi \leq x \leq \pi$ is:

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos(kx) + b_k \sin(kx) \right]$$

"has the Fourier series"

Where

$$a_k = \langle f, \cos(kx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx$$

$$b_k = \langle f, \sin(kx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx$$

Fourier Coefficients

Math 330

1. Find the Fourier series of $f(x) = |x|$. Then plot $f(x)$ together with partial sums of its Fourier series.

a_k: $\frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos(x)\, dx = \frac{2}{\pi}\int_{0}^{\pi} x \cdot \cos(x)\, dx$

since $|x| \cdot \cos(x)$ has even symmetry

b_k: $\frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin(x)\, dx = 0$ since $|x| \cdot \sin(x)$ has odd symmetry

2. Find the Fourier series of $f(x) = \begin{cases} 1, & \text{if } x \geq 0 \\ -1, & \text{if } x < 0 \end{cases}$. Then plot $f(x)$ together with partial sums of its Fourier series.

3. Find the Fourier series of $f(x) = 3x - 2$. Then plot $f(x)$ together with partial sums of its Fourier series.

4. What property of $f(x)$ guarantees that its Fourier series consists only of sine terms?

5. What property of $f(x)$ guarantees that its Fourier series consists of only cosine terms?

6. Based on the Fourier series that you have seen so far, make a conjecture about the points at which the Fourier series for $f(x)$ converges to $f(x)$.