Fourier Series

Math 330

1. Match the following $2\pi-$periodic functions with their Fourier series *without* solving for the coefficients.

We did this last time.

(a) $f(x) = x^2$ for $x \in [-\pi, \pi]$

(b) $f(x) = x\left(\pi^2 - x^2\right)$ for $x \in [-\pi, \pi]$

(c) $f(x) = \sin\left(\dfrac{x}{2}\right)$ for $x \in [-\pi, \pi]$

(d) $f(x) = \begin{cases} \frac{\pi}{2} + x, & -\pi \le x < 0 \\ \frac{\pi}{2} - x, & 0 \le x \le \pi \end{cases}$

I. $f(x) = \dfrac{8}{\pi} \displaystyle\sum_{k=1}^{\infty} \dfrac{k\,(-1)^k}{1-4k^2} \sin(kx)$

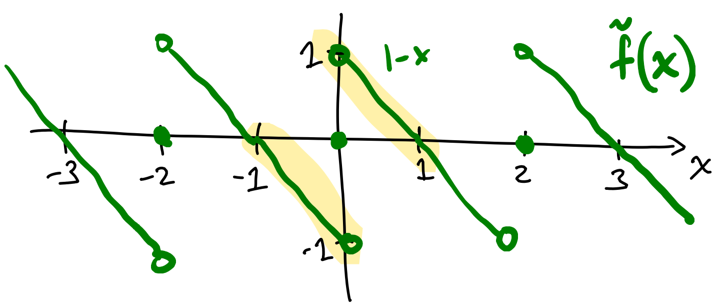
II. $f(x) = \dfrac{\pi^2}{3} + 4 \displaystyle\sum_{k=1}^{\infty} \dfrac{(-1)^k}{k^2} \cos(kx)$

III. $f(x) = -12 \displaystyle\sum_{k=1}^{\infty} \dfrac{(-1)^k}{k^3} \sin(kx)$

IV. $f(x) = \dfrac{2}{\pi} \displaystyle\sum_{k=1}^{\infty} \dfrac{1-(-1)^k}{k^2} \cos(kx)$

2. Let $f(x) = 1 - x$ be defined on $x \in [0, 1]$.

(a) Sketch $\tilde{f}(x)$, the odd 2-periodic extension of $f(x)$. That is, $\tilde{f}(x)$ should be an odd function with period 2.



(b) If you were to write a trigonometric series that converges $\tilde{f}(x)$, what terms would you put in this series?

odd

$\sin(\pi x)$

$\sin(2\pi x)$

$\vdots$

$\sin(k\pi x)$ for $k \in \mathbb{Z}$

even

$\cancel{\cos(\pi x)}$

$\cancel{\cos(2\pi x)}$

$\vdots$

$$\boxed{\tilde{f}(x) \sim \sum_{k=1}^{\infty} b_k \sin(k\pi x)}$$

$$\int_{-1}^{1} \tilde{f}(x) \sin(m\pi x)\,dx = \sum_{k=1}^{\infty} b_k \boxed{\int_{-1}^{1} \sin(k\pi x)\sin(m\pi x)\,dx}$$

zero, unless $k=m$

property: $\displaystyle\int_{-1}^{1} \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 0 & \text{if } m\neq n \text{ (integers)} \\ 1 & \text{if } m=n \text{ (integers)} \end{cases}$

(c) Compute the coefficients for your trigonometric series from part (b). Then plot some partial sums of this series together with $\tilde{f}(x)$.

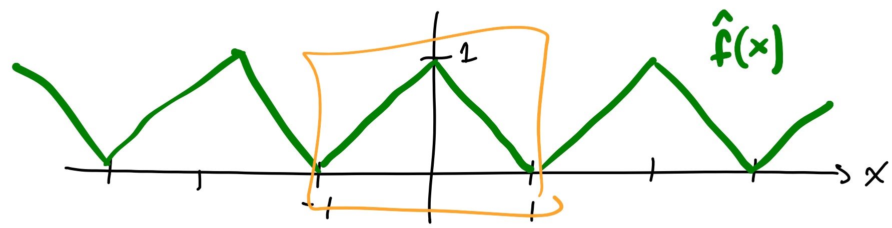
$$b_k = \int_{-1}^{1} \tilde{f}(x)\sin(k\pi x)\,dx = \int_{-1}^{0}(-x-1)\sin(k\pi x)\,dx + \int_{0}^{1}(1-x)\sin(k\pi x)\,dx$$

$$= 2\int_0^1 \underbrace{(1-x)}_{u}\,\underbrace{\sin(k\pi x)\,dx}_{dv} = \frac{2}{k\pi}$$

integrate by parts

$$\text{So: } \tilde{f}(x) = \sum_{k=1}^{\infty} \frac{2}{k\pi}\sin(k\pi x)$$

(d) Sketch $\hat{f}(x)$, the even 2-periodic extension of $f(x)$. That is, $\hat{f}(x)$ should be an even function with period 2.



(e) If you were to write a trigonometric series that converges $\hat{f}(x)$, what terms would you put in this series?

$$\hat{f}(x) \sim \sum_{k=0}^{\infty} a_k \cos(k\pi x)$$

(f) Compute the coefficients for your trigonometric series from part (e). Then plot some partial sums of this series together with $\hat{f}(x)$.

$$a_k = \int_{-1}^{1} \hat{f}(x)\cos(k\pi x)\,dx = 2\int_0^1 (1-x)\cos(k\pi x)\,dx = \frac{2(1-(-1)^k)}{k^2\pi^2} \quad \text{if } k\neq 0$$

$$a_0 = \int_{-1}^{1}\hat{f}(x)\,dx = 1$$

avg. value

$$\hat{f}(x) = \boxed{\frac{a_0}{2}} + \sum_{k=1}^{\infty} a_k\cos(k\pi x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{2(1-(-1)^k)}{k^2\pi^2}\cos(k\pi x)$$

More Fourier Series

Math 330

1. Use Euler's formula to express e^{ikx} in terms of sine and cosine functions. Then do the same for e^{-ikx}.

$$\boxed{e^{ix} = \cos x + i \sin x} \qquad i^2 = -1$$

$$e^{ikx} = \cos(kx) + i\sin(kx)$$

$$e^{-ikx} = \cos(-kx) + i\sin(-kx)$$

$$e^{-ikx} = \cos(kx) - i\sin(kx)$$

$e^{ikx} + e^{-ikx} = 2\cos(kx)$

ADD

2. Now express $\cos(kx)$ in terms of complex exponentials. Then do the same for $\sin(kx)$.

$$\cos(kx) = \frac{e^{ikx} + e^{-ikx}}{2} \qquad e^{ikx} - e^{-ikx} = 2i\sin(kx)$$

$$\sin(kx) = \frac{e^{ikx} - e^{-ikx}}{2i}$$

3. Use your formulas to convert a trigonometric Fourier series into a series of complex exponentials. Specifically, fill in the boxes in the following equation.

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k\cos(kx) + b_k\sin(kx)\right) = \boxed{\frac{a_0}{2}e^0} + \sum_{k=1}^{\infty}\left(\boxed{\frac{a_k+ib_k}{2}}\,e^{-ikx} + \boxed{\frac{a_k-ib_k}{2}}\,e^{ikx}\right)$$

$$= \frac{a_0}{2}e^0 + \sum_{k=1}^{\infty}\left[a_k\frac{e^{ikx}+e^{-ikx}}{2} + b_k\frac{e^{ikx}-e^{-ikx}}{2i}\right]$$

$$= \frac{a_0}{2}e^0 + \sum_{k=1}^{\infty}\left[\left(\frac{a_k}{2} + \frac{b_k i}{2}\right)e^{-ikx} + \left(\frac{a_k}{2} - \frac{b_k i}{2}\right)e^{ikx}\right]$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{+ikx}$$

where: $c_{-k} = \frac{a_k+ib_k}{2}$ (neg)

$c_0 = \frac{a_0}{2}$ (0)

$c_k = \frac{a_k-ib_k}{2}$ (pos)

4. If $f(x) \sim \sum_{k=-\infty}^{\infty} c_k e^{ikx}$ on $[-\pi,\pi]$, then the complex Fourier coefficients are computed via

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}\,dx.$$

$$\left[\int_{-\pi}^{\pi} e^{imx}e^{-inx}\,dx = 0 \quad \text{unless } m=n\right]$$

Compute the complex Fourier coefficients for $f(x) = e^x$. Then plot $f(x)$ together with partial sums of its complex Fourier series.

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-ikx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(1-ik)x}\,dx = \frac{1}{2\pi}\left[\frac{1}{1-ik}e^{(1-ik)x}\right]_{x=-\pi}^{\pi}$$

$$e^{-ik} = (e^{-i})^k$$

$$= \frac{1}{2\pi}\cdot\frac{1}{(1-ik)}\left[e^{\overset{\pi-i\pi k}{\pi(1-ik)}} - e^{-\pi(1-ik)}\right] = \frac{e^{\pi}(e^{-i\pi})^k - e^{-\pi}(e^{i\pi})^k}{2\pi(1-ik)} = \frac{e^{\pi}(-1)^k - e^{-\pi}(-1)^k}{2\pi(1-ik)}$$

$$e^{i\pi} = -1 = e^{-i\pi}$$

$$e^{i\pi} = \overset{-1}{\cos\pi} + i\,\overset{0}{\cancel{\sin\pi}}$$

To be continued next time...

5. Let $\hat{f}(x)$ be the 2π-periodic even extension of e^x. Sketch $\hat{f}(x)$.

6. Compute the Fourier series of $\hat{f}(x)$. Plot some partial sums of this Fourier series.

7. What happens if you differentiate the Fourier series for $\hat{f}(x)$ term by term to obtain $\hat{f}'(x)$? Plot some partial sums of $\hat{f}'(x)$.

8. What happens if you differentiate the Fourier series for $\hat{f}'(x)$ term by term to obtain $\hat{f}''(x)$? Plot some partial sums of $\hat{f}''(x)$.