# OVERVIEW

MATH 330
12 October 2023

## WAVES (Ch. 2)

Transport: $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$

Wave Eq: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

method of characteristics $\xi = x + ct$

d'Alembert's solution

— traveling waves —

## FOURIER SERIES (ch. 3)

$$f(x) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k \cos(kx) + b_k \sin(kx)\right)$$

## HEAT EQUATION: (ch. 4)

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}$$

with: initial conditions (ICs)
boundary conditions (BCs)

Separation of variables: $u(t,x) = G(t)\,v(x)$

we find: $\frac{G'(t)}{\gamma G(t)} = \frac{v''(x)}{v(x)} = -\lambda$

$$G' = -\lambda\gamma G$$

$$G(t) = e^{-\lambda\gamma t}$$

$$\underbrace{v'' = -\lambda v}$$

2 ODEs

with boundary conditions

BVP gives eigenvalues $\lambda_k$ and eigenfunctions $\cos(kx)$, $\sin(kx)$ for nonneg. integers $k$

general solution to heat eq:

$$u(t,x) = \sum_{k=1}^{\infty} e^{-\lambda_k \gamma t}\left(a_k \cos(kt) + b_k \sin(kt)\right)$$

Then use the initial conditions

$u(0,x) = f(x)$ to solve for the

coefficients $a_k$, $b_k$,

thus finding the particular solution.

# Separation of Variables

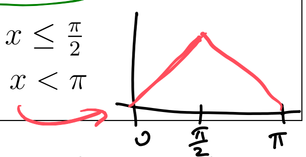
Math 330

**Problem I**: heat equation with homogeneous Dirichlet boundary conditions (i.e., zero temperature at endpoints)


PDE: $$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}$$

Boundary Conditions: $u(t,0) = 0$ and $u(t,\pi) = 0$

Initial Condition: $$u(0,x) = \begin{cases} x, & 0 < x \le \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases}$$




**1.** We will look for solutions of the form $u(t,x) = G(t)v(x)$, where $G = G(t)$ is a function of $t$ only and $v = v(x)$ is a function of $x$ only. *Plug this solution into the PDE and separate variables*: move everything that depends on $t$ to the left side of the equation, and everything that depends on $x$ to the right side. As good practice, keep the derivative expressions in the numerators, and group the $\gamma$ constant with the $G$ function.

$$\frac{\partial u}{\partial t} = G'(t)\,v(x) \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = G(t)\,v''(x)$$

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2} \quad \text{becomes} \quad G'(t)\,v(x) = \gamma\, G(t)\, v''(x)$$

or: $$\frac{G'(t)}{\gamma G(t)} = \frac{v''(x)}{v(x)}$$

function of $t$ function of $x$ alone

**2.** Each side of your new equation equals a constant, in fact the same constant. *Why is this?*

$$\frac{G'(t)}{\gamma G(t)} = \frac{v''(x)}{v(x)} = \text{some constant} = -\lambda$$

**3.** Denote the constant by $-\lambda$ (the negative is just for convenience later). Since each side of the equation equals $-\lambda$, this produces two ordinary differential equations, one in $G(t)$ and the other in $v(x)$. Write down these two ordinary differential equations. Simplify or rearrange them so that they look familiar enough to solve.

$$G'(t) = -\lambda\gamma G(t) \qquad v''(x) = -\lambda v(x)$$

exponential decr

**4.** Solve the time-dependent ODE for $G(t)$. Of the three options, $\boxed{\lambda > 0}$, $\boxed{\lambda = 0}$, or ~~$\lambda < 0$~~, which ones seem the most physically relevant?

constant

$$G'(t) = -\lambda \gamma G(t)$$

has solution $\boxed{G(t) = c \cdot e^{-\lambda \gamma t}}$

Assume $c \neq 0$

**5.** The position-dependent ODE, along with the given boundary conditions, form a boundary value problem for $v(x)$. Find the general solution for each of the three cases, $\lambda > 0$, $\lambda = 0$, and $\lambda < 0$. Which cases yield a nontrivial solution?

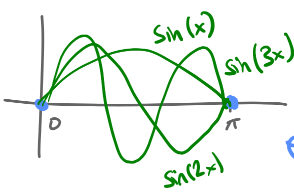
boundary:

$v''(x) = -\lambda v(x)$

$u(t,0) = 0$ $\quad$ $u(t,\pi) = 0$

$G(t)v(0) = 0$

$v(0) = 0$ since $G \neq 0$ $\quad$ $v(\pi) = 0$




eigenvalues $\lambda_n = n^2$

eigenfunctions $v_n(x) = \sin(nx)$

for $n = 1, 2, 3, \ldots$

$u(t,x) = e^{-\gamma \lambda t} \sin(nx)$ for $n = 1, 2, 3, \ldots$

$$\boxed{\text{General solution to the PDE: } u(t,x) = \sum_{n=1}^{\infty} b_n e^{-\gamma n^2 t} \sin(nx)}$$

set $t=0$: $u(0,x) = \sum_{n=1}^{\infty} b_n e^{0} \sin(nx)$

**6.** You now have a general solution to Problem 1, using everything except the initial condition. Finally, use the initial condition to obtain a particular solution.

$$u(0,x) = f(x) = \begin{cases} x, & 0 < x < \frac{\pi}{2} \\ \pi - x, & \frac{\pi}{2} < x < \pi \end{cases} \quad \text{and} \quad f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

Solve for $b_n$: $\quad b_n = \int_0^{\pi} f(x) \sin(nx)\, dx = \dfrac{4 \cdot \sin\left(\frac{n\pi}{2}\right)}{n^2 \pi}$

$$\boxed{\text{Particular Solution: } u(t,x) = \sum_{n=1}^{\infty} \frac{4 \sin\left(\frac{n\pi}{2}\right)}{n^2 \pi} e^{-\gamma n^2 t} \sin(nx)}$$

**7.** Use technology to plot your particular solution over time. (Set $\gamma = 1$.) Does it seem reasonable to you?

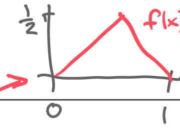
For "large" $t$: $\quad u(t,x) \approx \underbrace{\frac{4}{\pi} e^{-\gamma t} \sin(x)}_{n=1 \text{ term}}$

**Problem II**: heat equation with homogeneous Neumann boundary conditions (i.e., zero flux at endpoints)


PDE: $$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}$$

Boundary Conditions: insulated endpoints $$\boxed{\frac{\partial u}{\partial x}(t,0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(t,1) = 0}$$

Initial Condition: $$u(0,x) = \begin{cases} x, & 0 < x \le \frac{1}{2} \\ 1-x, & \frac{1}{2} < x < 1 \end{cases}$$




Work through the steps on the previous page to solve Problem II.

Suppose $u(t,x) = G(t)\,v(x)$.

Then $\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}$ becomes $G'(t)v(x) = \gamma\, G(t)\, v''(x)$.

Separating variables, we find $\frac{G'(t)}{\gamma G(t)} = \frac{v''(x)}{v(x)} = -\lambda$, for some constant $\lambda$

This implies: $G'(t) = -\lambda \gamma G(t)$ and $\boxed{v''(x) = -\lambda v(x).}$ ← boundary-value problem

$\Downarrow$

$G(t) = e^{-\lambda \gamma t}$

Boundary conditions: $\frac{\partial u}{\partial x}(t,0) = G(t)\,v'(0) = 0 \Rightarrow \boxed{v'(0)=0}$

$\frac{\partial u}{\partial x}(t,1) = G(t)\,v'(1) = 0 \Rightarrow \boxed{v'(1)=0}$

- If $\lambda < 0$, then $v(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$, and only the trivial solution $c_1 = c_2 = 0$ satisfies the boundary conditions
- If $\lambda = 0$, then $v(x) = c_1 x + c_2$. The boundary conditions imply $c_1 = 0$, so we have constant solutions $v(x) = c$.
- If $\lambda > 0$, then $v(x) = c_1 \cos(\sqrt{\lambda}\,x) + c_2 \sin(\sqrt{\lambda}\,x)$. The boundary conditions imply $c_2 = 0$ and $\lambda = (n\pi)^2$ for any positive integer $n$.

Eigenvalues: $\lambda_n = (n\pi)^2$

Eigenfunctions: $v_n = \cos(n\pi x)$ for $n \in \{0, 1, 2, 3, \dots\}$

## General Solution:

$$u(t,x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\, e^{-\gamma (n\pi)^2 t} \cos(n\pi x)$$

Now use the initial condition to solve for the coefficients $a_n$:

$$u(0,x) = \frac{a_0}{2} + \sum_{n=0}^{\infty} a_n \cos(n\pi x) = f(x) = \begin{cases} x, & 0 < x \le \frac{1}{2} \\ 1-x, & \frac{1}{2} < x < 1 \end{cases}$$

Integrate:

$$a_0 = 2\int_0^1 f(x)\,dx = \frac{1}{2}$$

[Refer to page 96 in the text.]

$$a_n = 2\int_0^1 f(x)\cos(n\pi x)\,dx = \frac{8\cos\left(\frac{n\pi}{2}\right)\sin^2\left(\frac{n\pi}{4}\right)}{n^2\pi^2}$$

**Particular Solution:**

$$u(t,x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{8\cos\left(\frac{n\pi}{2}\right)\sin^2\left(\frac{n\pi}{4}\right)}{n^2\pi^2}\, e^{-\gamma n^2\pi^2 t} \cos(n\pi x)$$