MATH 330 19 October 2023

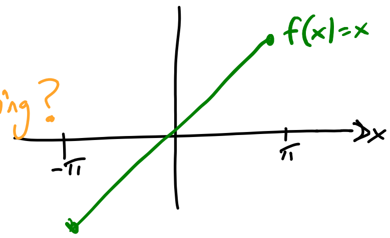
Problem III: heat equation with periodic boundary conditions (models an insulated circular ring)

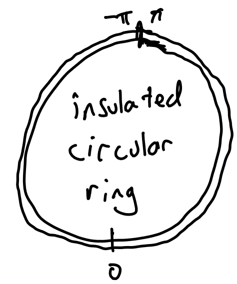
PDE: $\dfrac{\partial u}{\partial t} = \gamma \dfrac{\partial^2 u}{\partial x^2}$

Boundary Conditions: $u(t,-\pi) = u(t,\pi)$ and $\dfrac{\partial u}{\partial x}(t,-\pi) = \dfrac{\partial u}{\partial x}(t,\pi)$

Initial Condition: $f(x) = \underbrace{u(0,x) = x}$ for $-\pi < x < \pi$

Use the method of separation of variables to solve Problem III.





→ What is the equilibrium heat distribution in this setting?

satisfies $\dfrac{\partial u}{\partial t} = 0 = \gamma \dfrac{\partial^2 u}{\partial x^2}$

$\dfrac{\partial^2 u}{\partial x^2} = 0$ has solution $u^*(x) = c_1 x + c_2$

boundary conditions: $u^*(-\pi) = u^*(\pi)$ and $u^{*\prime}(-\pi) = u^{*\prime}(\pi)$

$c_1(-\pi) + \cancel{c_2} = c_1(\pi) + \cancel{c_2}$ $\qquad c_1 = c_1$

$-\pi c_1 = \pi c_1$

$c_1 = 0$ slope zero: $\boxed{u^*(x) = c_2}$

equilibrium solution is any constant

What is the general solution?

suppose $u(t,x) = G(t)\,v(x)$. As before, $G(t) = e^{-\gamma\lambda t}$.

Also: $v(x)$ satisfies $v''(x) = -\lambda v(x)$ with boundary conditions $v(-\pi) = v(\pi)$ and $v'(-\pi) = v'(\pi)$. BVP

- If $\lambda < 0$, then $v(x) = c_1 e^{\sqrt{-\lambda}\,x} + c_2 e^{-\sqrt{-\lambda}\,x}$, which cannot satisfy BCs.
- If $\lambda = 0$, then $v(x) = c_1 x + c_2$ and $c_1 = 0$, so $v(x) = c_2$ is constant solution.
- If $\lambda > 0$, then $v(x) = c_1 \cos(\sqrt{\lambda}\,x) + c_2 \sin(\sqrt{\lambda}\,x)$

$\sqrt{\lambda}x = kx$

this satisfies BCs if $\sqrt{\lambda}$ is an integer, so $\lambda = n^2$ for $n = 1, 2, 3, 4, \ldots$

Product solutions: $u(t,x) = \left(c_1 \cos(nx) + c_2 \sin(nx)\right) e^{-\gamma n^2 t}$

$$\boxed{\text{General Solution: } u(t,x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx)\right) e^{-\gamma n^2 t}}$$

At what rate does the temperature approach equilibrium?

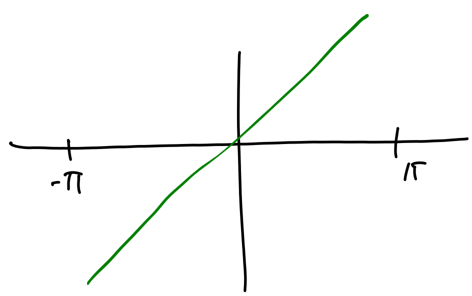
exponential rate given by smallest (nonzero) eigenvalue

$\lambda_1 = 1 \qquad e^{-\gamma t}$

What is the shape of the solution "just before" it reaches equilibrium?

$$\frac{a_0}{2} + a_1 \cos(x) + b_1 \sin(x)$$

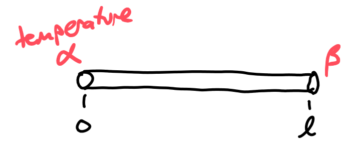
Initial condition $f(x) = x$, $-\pi < x < \pi$



initial condition is odd fn $\Rightarrow a_n = 0$

Heat Equation with Inhomogeneous Boundary Conditions

Math 330



1. Consider the following boundary-value problem:

PDE: $$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}$$

Boundary Conditions: $u(t,0) = \alpha$ and $u(t,\ell) = \beta$

(a) Let $u^\star(x)$ denote the equilibrium solution to this boundary-value problem. Find $u^\star(x)$.

$\frac{\partial u}{\partial t} = 0$

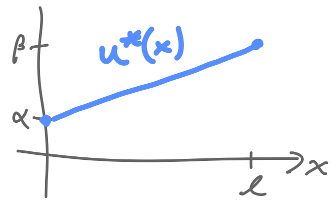
$$0 = \gamma \frac{\partial^2 u^*}{\partial x^2}$$

$$\Downarrow$$

$$u^*(x) = c_1 x + c_2$$

$$\underbrace{u^*(0) = \alpha, \quad u^*(\ell) = \beta}_{\text{BCs imply}}$$

$$c_2 = \alpha, \quad c_1 = \frac{\beta - \alpha}{\ell}$$



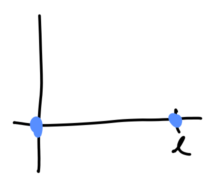
$$\boxed{u^*(x) = \frac{\beta-\alpha}{\ell} x + \alpha}$$

(b) Let $\underbrace{\tilde{u}(t,x)}_{\text{"transient solution"}} = \underbrace{u(t,x)}_{\text{general sol}} - \underbrace{u^\star(x)}_{\text{equilibrium sol}}$. What boundary conditions are satisfied by $\tilde{u}(t,x)$?

$$\left.\begin{aligned} \tilde{u}(t,0) &= u(t,0) - u^*(0) = \alpha - \alpha = 0 \\ \tilde{u}(t,\ell) &= u(t,\ell) - u^*(\ell) = \beta - \beta = 0 \end{aligned}\right\} \quad \tilde{u} \text{ satisfies homogeneous BCs}$$

(c) Write the general solution to the heat equation with the boundary conditions you identified in #2. *Hint*: $\underbrace{\text{You have already solved a very similar problem on a previous worksheet.}}_{\text{Problem I}}$

$$\tilde{u}(t,x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\gamma n^2 t}$$



(d) Write the general solution to the heat equation with the boundary conditions at the top of the page.

$$\boxed{u(t,x) = \alpha + \frac{\beta-\alpha}{\ell} x + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\ell}\right) e^{-\gamma n^2 t}}$$

2. Solve the following heat equation with inhomogeneous flux boundary conditions

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 1, \qquad u(0,x) = 1$$

by following the steps below.

(a) First, find the steady state solution $u^\star(x)$. Note that there is *not* an arbitrary constant in the steady state solution. Find the constant using the fact that heat is conserved in this problem; heat is entering the system on the left at exactly the same rate it is leaving the system on the right.

(b) Next, find the transient solution $\tilde{u}(t,x)$ that satisfies the PDE with homogeneous BCs using separation of variables. *Hint*: You have already solved this problem on a previous worksheet.

(c) Finally, use the fact that $u(t,x) = u^\star(x) + \tilde{u}(t,x)$ and the initial condition to find the solution to the PDE with inhomogeneous BCs.

(d) Plot the temperature profile, truncated to a reasonable number of terms, at several time points. Explain how the solution behaves as $t \to \infty$.

(a) Equilibrium solution satisfies $0 = \frac{\partial^2 u}{\partial x^2}$, $\frac{\partial u}{\partial x}(0) = 1$, and $\frac{\partial u}{\partial x}(1) = 1$

$0 = \frac{\partial^2 u}{\partial x^2}$ implies $u^*(x) = c_1 x + c_2$

boundary conditions:

$\frac{\partial u^*}{\partial x}(x) = c_1$, so $\frac{\partial u}{\partial x}(0) = 1$ implies $c_1 = 1$.

c_2 is not determined by the boundary conditions

Thus, $u^*(x) = x + c_2$.

BONUS: Since $\frac{\partial u}{\partial x}(0) = \frac{\partial u}{\partial x}(1)$, we have constant heat flux at the endpoints. Heat energy flowing out the left end is exactly balanced by heat energy flowing in the right end, so the amount of heat energy in the rod does not change. The initial heat energy equals the final heat energy, so the area under the initial condition equals the area under the equilibrium solution:

$$\int_0^1 1\,dx = \int_0^1 (x + c_2)\,dx$$

$$1 = \left[\tfrac{1}{2}x^2 + c_2\right]_0^1$$

$$1 = \tfrac{1}{2} + c_2$$

$$\tfrac{1}{2} = c_2$$

Therefore, $\boxed{u^*(x) = x + \frac{1}{2}.}$

(b) We want to solve: $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$ with $\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0$.

This was Problem II from last week, and the solution is:

$$\tilde{u}(t,x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)\exp\left(-\tfrac{1}{2}n^2\pi^2 t\right)$$

(c) Now $u(t,x) = u^*(x) + \tilde{u}(t,x)$

$$u(t,x) = \left(x+\tfrac{1}{2}\right) + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)\exp\left(-\tfrac{1}{2}n^2\pi^2 t\right)$$

Set $t=0$: $$1 = x + \tfrac{1}{2} + \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

initial condition

$$\underbrace{\tfrac{1}{2} - x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)}_{\text{Solve for the Fourier coefficients:}}$$

$$a_0 = 2\int_0^1 \left(\tfrac{1}{2} - x\right)dx = 0$$

$$a_n = 2\int_0^1 \left(\tfrac{1}{2} - x\right)\cos(n\pi x)\,dx = 2\,\frac{1-(-1)^n}{(n\pi)^2} \quad \text{for } n \in \mathbb{Z}^+$$

Therefore:

$$\boxed{u(t,x) = x + \tfrac{1}{2} + \sum_{n=1}^{\infty} 2\,\frac{1-(-1)^n}{n^2\pi^2}\cos(n\pi x)\exp\left(-\tfrac{1}{2}n^2\pi^2 t\right)}$$

(d) Plots: