Heat Equation with Inhomogeneous Boundary Conditions

Math 330

1. Consider the following boundary-value problem:

PDE:	$\dfrac{\partial u}{\partial t} = \gamma \dfrac{\partial^2 u}{\partial x^2}$
Boundary Conditions:	$u(t,0) = \alpha \quad \text{and} \quad u(t,\ell) = \beta$

(a) Let $u^\star(x)$ denote the equilibrium solution to this boundary-value problem. Find $u^\star(x)$.

(b) Let $\tilde{u}(t,x) = u(t,x) - u^\star(x)$. What boundary conditions are satisfied by $\tilde{u}(t,x)$?

(c) Write the general solution to the heat equation with the boundary conditions you identified in #2. *Hint*: You have already solved a very similar problem on a previous worksheet.

(d) Write the general solution to the heat equation with the boundary conditions at the top of the page.

2. Solve the following heat equation with inhomogeneous flux boundary conditions

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}, \qquad \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 1, \qquad u(0,x) = 1$$

by following the steps below.

(a) First, find the steady state solution $u^\star(x)$. Note that there is *not* an arbitrary constant in the steady state solution. Find the constant using the fact that heat is conserved in this problem; heat is entering the system on the left at exactly the same rate it is leaving the system on the right.

(b) Next, find the transient solution $\tilde{u}(t,x)$ that satisfies the PDE with homogeneous BCs using separation of variables. *Hint*: You have already solved this problem on a previous worksheet.

(c) Finally, use the fact that $u(t,x) = u^\star(x) + \tilde{u}(t,x)$ and the initial condition to find the solution to the PDE with inhomogeneous BCs.

(d) Plot the temperature profile, truncated to a reasonable number of terms, at several time points. Explain how the solution behaves as $t \to \infty$.