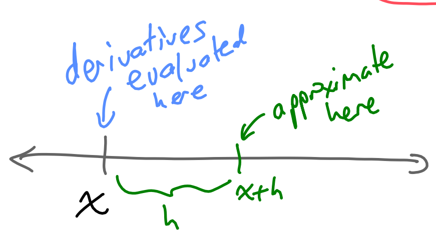
Finite Difference Approximations

How can we approximate a function with a polynomial?

Taylor's Theorem: If $f(x)$ has $k+1$ derivatives, then

$$f(x+h) = \underbrace{f(x) + f'(x)\cdot h + \tfrac{1}{2}f''(x)h^2 + \cdots + \tfrac{1}{k!}f^{(k)}(x)\,h^k}_{\text{Taylor polynomial of degree } k} + \underbrace{\tfrac{1}{(k+1)!}f^{(k+1)}(\xi)\,h^{k+1}}_{\text{error term}}$$



error term, for some ξ, $x \le \xi \le x+h$

Example: Difference Quotient

Suppose $u(x)$ is a C^2 functions. Then:

$$\boxed{u(x+h) = \underbrace{u(x) + u'(x)\cdot h}_{\text{Taylor polynomial}}} + \underbrace{\tfrac{1}{2}u''(\xi)h^2}_{\text{error term}}$$

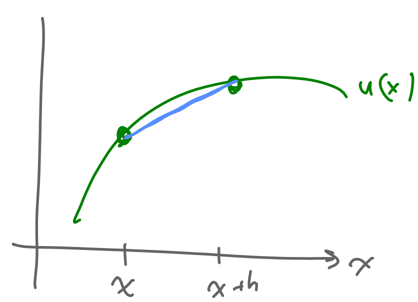
Then:

$$\boxed{u'(x) \approx \underbrace{\frac{u(x+h) - u(x)}{h}}_{\text{difference quotient}}}$$

with truncation error $\tfrac{1}{2}u''(\xi)\cdot h$

FORWARD DIFFERENCE APPROXIMATION

difference quotient → slope of the secant line



Let $C = \max\limits_{x\in[x,x+h]} \left|\tfrac{1}{2}u''(x)\right|$, then the truncation error is $\le C|h|$

First-order approximation.

We say the error is $\underbrace{O(h^1)}_{\text{"big-O of h"}}$, which means the error is at worst a constant multiple of h.

Derivative Approximations

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1. Let u be a C^3 function.

(a) Write the Taylor polynomial of degree 2 for $u(x+h)$.

$$u(x+h) = u(x) + u'(x)h + \tfrac{1}{2}u''(x)h^2 + \tfrac{1}{6}u'''(\xi)h^3$$

$O(h^3)$

(b) Write the Taylor polynomial of degree 2 for $u(x-h)$.

$$u(x-h) = u(x) - u'(x)h + \tfrac{1}{2}u''(x)h^2 - \tfrac{1}{6}u'''(\xi)h^3$$

(c) Subtract one Taylor polynomial from the other, and solve for $u'(x)$. You now have an approximation for the first derivative. What is the order of its truncation error?

$$u(x+h) - u(x-h) = 2u'(x)h + O(h^3)$$

the error term here is not more than a multiple of h^3

CENTERED DIFFERENCE APPROXIMATION

$$\boxed{u'(x) = \frac{u(x+h) - u(x-h)}{2h}} + O(h^2)$$

second-order approximation

2. Let u be a C^4 function.

(a) Write the Taylor polynomial of degree 3 for $u(x+h)$.

$$u(x+h) = u(x) + u'(x)h + \tfrac{1}{2}u''(x)h^2 + \tfrac{1}{6}u'''(x)h^3 + O(h^4)$$

(b) Write the Taylor polynomial of degree 3 for $u(x-h)$.

$$u(x-h) = u(x) - u'(x)h + \tfrac{1}{2}u''(x)h^2 - \tfrac{1}{6}u'''(x)h^3 + O(h^4)$$

(c) Add the two Taylor polynomials and solve for $u''(x)$. You now have an approximation for the second derivative. What is the order of its truncation error?

$$u(x+h) + u(x-h) = 2u(x) + u''(x)h^2 + O(h^4)$$

$$\boxed{u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}}$$

with error $O(h^2)$

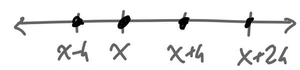
Centered Difference Approximation of $u''(x)$

3. Here is one more example that suggests a more general approach for finding finite difference approximations to derivatives. We seek an approximation of $u'(x)$ using the values $u(x-h)$, $u(x)$, $u(x+h)$, and $u(x+2h)$.

(a) Write the Taylor polynomials of degree 3 for $u(x-h)$, $u(x)$, $u(x+h)$, and $u(x+2h)$.

$$u(x-h) = 1u(x) - 1u'(x)h + \tfrac{1}{2}u''(x)h^2 - \tfrac{1}{6}u'''(x)h^3 \quad + O(h^4)$$

$h=0$: $\quad u(x) = u(x) \quad \leftarrow$ no error



$$u(x+h) = u(x) + u'(x)h + \tfrac{1}{2}u''(x)h^2 + \tfrac{1}{6}u'''(x)h^3 \quad + O(h^4)$$

$$u(x+2h) = u(x) + u'(x)\,2h + \tfrac{1}{2}u''(x)(2h)^2 + \tfrac{1}{6}u'''(x)(2h)^3 \quad + O(h^4)$$

(b) View your Taylor polynomials as a system of linear equations of the following form:

$$\begin{bmatrix} u(x-h) \\ u(x) \\ u(x+h) \\ u(x+2h) \end{bmatrix} = A \begin{bmatrix} u(x) \\ u'(x)h \\ u''(x)h^2 \\ u'''(x)h^3 \end{bmatrix}$$

where A is a 4×4 coefficient matrix. What is this matrix?

$$\begin{bmatrix} u(x-h) \\ u(x) \\ u(x+h) \\ u(x+2h) \end{bmatrix} = \begin{bmatrix} 1 & -1 & \frac{1}{2} & \frac{-1}{6} \\ 1 & 0 & 0 & 0 \\ 1 & 1 & \frac{1}{2} & \frac{1}{6} \\ 1 & 2 & 2 & \frac{4}{3} \end{bmatrix} \begin{bmatrix} u(x) \\ u'(x)h \\ u''(x)h^2 \\ u'''(x)h^3 \end{bmatrix} \quad + O(h^4)$$

(c) Find A^{-1}. (Use technology.) Use the entries in the second row of A^{-1} to write down an approximation of $u'(x)$. What is the order of the truncation error?

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{-1}{3} & \frac{-1}{2} & 1 & \frac{-1}{6} \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} u(x-h) \\ u(x) \\ u(x+h) \\ u(x+2h) \end{bmatrix} = \begin{bmatrix} u(x) \\ u'(x)h \\ u''(x)h^2 \\ u'''(x)h^3 \end{bmatrix} \quad + O(h^4)$$

$$u'(x) = \frac{1}{h}\left(\frac{-1}{3}u(x+h) - \frac{1}{2}u(x) + u(x+h) - \frac{1}{6}u(x+2h)\right) \quad + O(h^3)$$

(d) Note that the entries of A^{-1} also give you approximations of $u''(x)$ and $u'''(x)$.

for example: $\quad u'''(x) = \frac{1}{h^3}\left(-u(x-h) + 3u(x) - 3u(x+h) + u(x+2h)\right) \quad + O(h)$