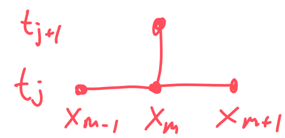
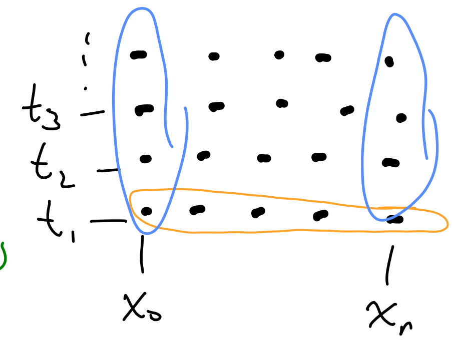
Heat Equation $\underbrace{\text{Explicit Scheme}}_{\text{forward difference in time}}$

$$u_{j,m} = u(t_j, x_m)$$



$$\boxed{u_{j+1,m} = \mu\, u_{j,m+1} + (1-2\mu)\, u_{j,m} + \mu\, u_{j,m-1}}$$



Stability Analysis:

If we let $u_{j,m} = e^{ikx_m}$, then we find

$$u_{j+1,m} = \lambda \cdot u_{j,m}$$

If $\mu = \frac{\gamma \Delta t}{(\Delta x)^2} < \frac{1}{2}$, then $|\lambda| < 1$, and the solution is <u>stable</u>.

If $\mu > \frac{1}{2}$, then $|\lambda| > 1$, and the solution is <u>unstable</u>.

Implicit Scheme for the Heat Equation

Math 330

Consider the heat equation with homogeneous Dirichlet boundary conditions:

$$\frac{\partial u}{\partial t} = \gamma \frac{\partial^2 u}{\partial x^2}, \qquad u(t,0) = 0, \qquad u(t,\ell) = 0, \qquad u(0,x) = f(x)$$

We will derive a partial difference equation using the backward difference in time. This results in the *implicit scheme* for approximating solutions to the heat equation.

error $O(\Delta t)$ error $O(\Delta x^2)$

1. Substitute a backward difference for $\frac{\partial u}{\partial t}$ and a centered difference for $\frac{\partial^2 u}{\partial x^2}$ into heat equation. Let $u_{j,m} = u(t_j, x_m)$. Write your equation in the form

$$u_{j-1,m} = \cancel{u_{m-1,j}} = \cdots$$

$$\frac{u(t_j, x_m) - u(t_{j-1}, x_m)}{\Delta t} = \gamma \frac{u(t_j, x_{m+1}) - 2u(t_j, x_m) + u(t_j, x_{m-1})}{(\Delta x)^2}$$

$$u_{j,m} - u_{j-1,m} = \underbrace{\frac{\gamma \Delta t}{(\Delta x)^2}}_{\mu}\left(u_{j,m+1} - 2u_{j,m} + u_{j,m-1}\right)$$

$$u_{j-1,m} = u_{j,m} - \mu\left(u_{j,m+1} - 2u_{j,m} + u_{j,m-1}\right)$$

$$\boxed{u_{j-1,m} = -\mu\, u_{j,m-1} + (1+2\mu)\, u_{j,m} - \mu\, u_{j,m+1}}$$

2. Write your partial difference equation as a vector equation:

$$\mathbf{u}^{(j-1)} = A\mathbf{u}^{(j)}$$

where $\mathbf{u}^{(j)}$ is the vector whose mth entry is $u_{j,m}$.

$$\begin{bmatrix} u_{j-1,1} \\ u_{j-1,2} \\ u_{j-1,3} \\ \vdots \\ u_{j-1,n-1} \end{bmatrix} = \begin{bmatrix} 1+2\mu & -\mu & 0 & \cdots & 0 \\ -\mu & 1+2\mu & -\mu & \cdots & 0 \\ 0 & -\mu & 1+2\mu & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1+2\mu \end{bmatrix} \begin{bmatrix} u_{j,1} \\ u_{j,2} \\ u_{j,3} \\ \vdots \\ u_{j,n-1} \end{bmatrix}$$

3. Download the Mathematica file for the implicit scheme from the course website. Complete the specification of matrix A in the code. Run the code for several choices of step sizes and initial conditions. What do you observe?

This method appears to give reasonable solutions without crazy oscillations or divergence.

4. Perform a stability analysis of the implicit scheme. That is, let $u_{j,m} = e^{ikx_m}$ and simplify the difference equation as much as possible. Find a value λ such that $u_{j,m} = \lambda u_{j-1,m}$. Under what conditions is $|\lambda| \leq 1$?

Let $u_{j,m} = e^{ikx_m}$. Substituting this into the finite difference equation:

$$
\begin{aligned}
u_{j-1,m} &= -\mu e^{ikx_{m-1}} + (1+2\mu) e^{ikx_m} - \mu e^{ikx_{m+1}} \\
&= -\mu e^{ik(x_m - \Delta x)} + (1+2\mu) e^{ikx_m} - \mu e^{ik(x_m + \Delta x)} \\
&= e^{ikx_m} \left[-\mu \left(e^{ik\Delta x} + e^{-ik\Delta x} \right) + 1 + 2\mu \right] \\
&= u_{j,m} \left[-2\mu \cos(k\Delta x) + 1 + 2\mu \right] \\
&= u_{j,m} \left[1 + 2\mu \left(1 - \cos(k\Delta x) \right) \right]
\end{aligned}
$$

Thus, $u_{j+1,m} = \lambda \cdot u_{j,m}$, where $\lambda = \left[1 + 2\mu \left(1 - \cos(k\Delta x) \right) \right]^{-1}$.

Stability is determined by $|\lambda|$.

Since $\mu > 0$ and $1 - \cos(k\Delta x) \geq 0$,

$$1 + 2\mu \left(1 - \cos(k\Delta x) \right) \geq 1.$$

Thus, $\lambda = \dfrac{1}{1 + 2\mu(1 - \cos(k\Delta x))} \leq 1$.

Therefore, the solution converges for all values μ, and the scheme is said to be unconditionally stable

Crank-Nicolson Scheme for the Heat Equation

Math 330

Consider the explicit (forward) and implicit (backward) schemes for the heat equation:

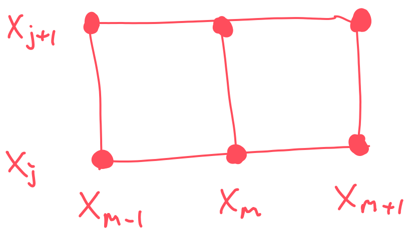
$$\text{explicit:} \qquad \frac{u_{j+1,m} - u_{j,m}}{\Delta t} = \gamma \frac{u_{j,m+1} - 2u_{j,m} + u_{j,m-1}}{(\Delta x)^2}$$

$$\text{implicit:} \qquad \frac{u_{j+1,m} - u_{j,m}}{\Delta t} = \gamma \frac{u_{j+1,m+1} - 2u_{j+1,m} + u_{j+1,m-1}}{(\Delta x)^2}$$

1. Add the two difference equations above and obtain a new difference equation of the form

$$u_{j+1,m} - u_{j,m} = \frac{\mu}{2}\Big[\qquad ? \qquad \Big]$$

Stencil diagram:



Let $\mu = \dfrac{\gamma \Delta t}{(\Delta x)^2}$. Then:

$$u_{j+1,m} - u_{j,m} = \frac{\mu}{2}\Big[u_{j+1,m+1} - 2u_{j+1,m} + u_{j+1,m-1} + u_{j,m+1} - 2u_{j,m} + u_{j,m-1}\Big]$$

errors: $O(\Delta t^2)$ and $O(\Delta x^2)$

2. Collect all the terms with time index $j+1$ on the left, and all the terms with time index j on the right. Then write your equation in vector form $A\mathbf{u}^{(j+1)} = B\mathbf{u}^{(j)}$ for some matrices A and B.

$$-\frac{\mu}{2}u_{j+1,m-1} + (1+\mu)u_{j+1,m} - \frac{\mu}{2}u_{j+1,m+1} = \frac{\mu}{2}u_{j,m-1} + (1-\mu)u_{j,m} + \frac{\mu}{2}u_{j,m+1}$$

$$\underbrace{\begin{bmatrix} 1+\mu & -\frac{\mu}{2} & & \\ -\frac{\mu}{2} & 1+\mu & \ddots & \\ & \ddots & \ddots & \\ & & & 1+\mu \end{bmatrix}}_{A} u^{(j+1)} = \underbrace{\begin{bmatrix} 1-\mu & \frac{\mu}{2} & & \\ \frac{\mu}{2} & 1-\mu & \ddots & \\ & \ddots & \ddots & \\ & & & 1-\mu \end{bmatrix}}_{B} u^{(j)}$$

COMPUTATION:

$u^{(j+1)} = A^{-1}B\,u^{(j)}$

3. Download the Mathematica file for the Crank-Nicolson scheme from the course website. Complete the specification of matrices A and B in the code. Run the code for several choices of step sizes and initial conditions. What do you observe?

The solution seems mostly reasonable, but there are some incorrect oscillations near where the initial condition has sharp corners.

4. Perform a stability analysis of the implicit scheme. It may help to let $u_{j,m} = e^{ikx_m}\lambda^j$ and solve for λ. Under what conditions is $|\lambda| \leq 1$?

Let $u_{j,m} = e^{ikx_m} \cdot \lambda^j$. Substituting into the finite difference equation:

$$e^{ikx_m}\lambda^{j+1} - e^{ikx_m}\lambda^j = \frac{\mu}{2}\left[e^{ikx_{m+1}}\lambda^j - 2e^{ikx_m}\lambda^j + e^{ikx_{m-1}}\lambda^j + e^{ikx_{m+1}}\lambda^{j+1} - 2e^{ikx_m}\lambda^{j+1} + e^{ikx_{m-1}}\lambda^{j+1}\right]$$

$$\cancel{e^{ikx_m}}\cancel{\lambda^j}(\lambda - 1) = \frac{\mu}{2}\cancel{e^{ikx_m}}\cancel{\lambda^j}\left[\underline{e^{ik\Delta x}} - 2 + \underline{e^{-ik\Delta x}} + \lambda \underline{e^{ik\Delta x}} - 2\lambda + \lambda \underline{e^{-ik\Delta x}}\right]$$

$$\lambda - 1 = \frac{\mu}{2}\left[2\cos(k\Delta x) - 2 + \lambda \cdot 2\cos(k\Delta x) - 2\lambda\right]$$

$$\lambda - 1 = \mu\left[\cos(k\Delta x) - 1 + \lambda\left(\cos(k\Delta x) - 1\right)\right]$$

$$\lambda - \lambda\mu\left(\cos(k\Delta x) - 1\right) = \mu\left(\cos(k\Delta x) - 1\right) + 1$$

Solve for λ:

$$\lambda = \frac{1 + \mu\left(\cos(k\Delta x) - 1\right)}{1 - \mu\left(\cos(k\Delta x) - 1\right)} = \frac{1 - \mu\left(1 - \cos(k\Delta x)\right)}{1 + \mu\left(1 - \cos(k\Delta x)\right)}$$

Since $\mu > 0$, we see that $|\lambda| < 1$ for all μ, so the Crank-Nicolson scheme is <u>unconditionally stable</u>.