TRANSPORT EQUATION

14 Nov. 2023
MATH 330

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

transport speed

Describes flow in a 1-dimensional medium, with constant speed c.

Solutions: $f(x-ct)$ for any C^1 function f.

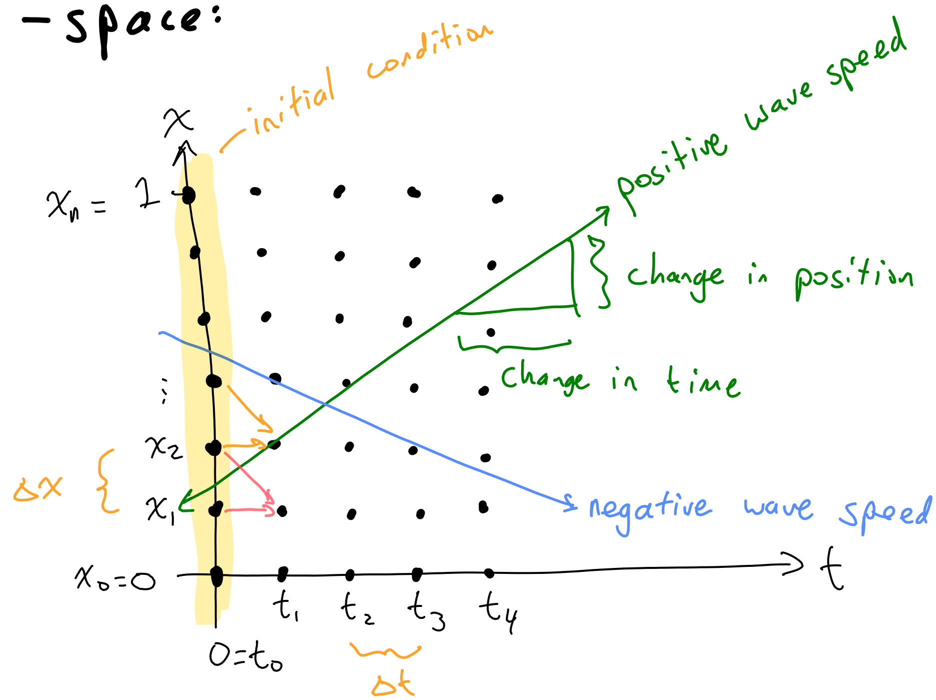
Why do numerical approximation for this equation?

We want to understand how to apply finite differences to this equation.

Finite differences are easier to generalize than exact solutions.

Transport eq. is a warm-up case for the wave equation and more general hyperbolic PDEs.

t, x -space:



initial condition
positive wave speed
change in position
change in time
negative wave speed
x
$x_n = 1$
x_2
Δx
x_1
$x_0 = 0$
t_1 t_2 t_3 t_4
t
$0 = t_0$
Δt

Finite Differences for the Transport Equation

Math 330

Recall the transport equation

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \qquad u(0,x) = f(x), \qquad -\infty < x < \infty.$$

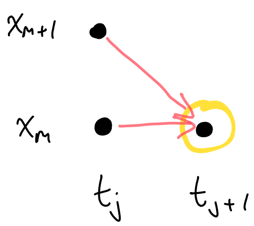
1. Use forward difference approximations to convert the transport equation into a finite difference equation. Let $u_{j,m} = u(t_j, x_m)$ and write your equation in the form

$$u_{j+1,m} =$$

where the right side of the equation involves $u_{j,m}$ and $u_{j,m+1}$.

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \frac{u_{j+1,m} - u_{j,m}}{\Delta t} + c\,\frac{u_{j,m+1} - u_{j,m}}{\Delta x} = 0$$

$$u_{j+1,m} - u_{j,m} = \frac{-c\,\Delta t}{\Delta x}\left(u_{j,m+1} - u_{j,m}\right)$$



Let $\sigma = \dfrac{c\,\Delta t}{\Delta x}$. Then:

$$\boxed{u_{j+1,m} = (\sigma + 1)\,u_{j,m} - \sigma\, u_{j,m+1}}$$

2. Write your partial difference equation as a vector equation:

$$\mathbf{u}^{(j+1)} = A\mathbf{u}^{(j)}$$

where $\mathbf{u}^{(j)}$ is the vector whose mth entry is $u_{j,m}$.

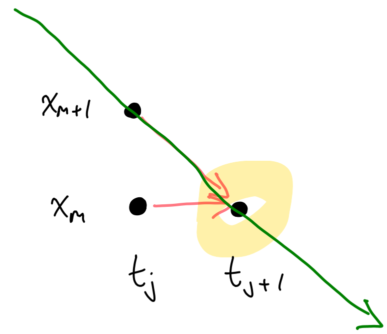
$$\begin{bmatrix} u_{j+1,0} \\ u_{j+1,1} \\ u_{j+1,2} \\ \vdots \\ u_{j+1,n} \end{bmatrix} = \begin{bmatrix} \sigma+1 & -\sigma & 0 & 0 & \cdots & 0 \\ 0 & \sigma+1 & -\sigma & 0 & \cdots & 0 \\ 0 & 0 & \sigma+1 & -\sigma & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ & & & & & -\sigma \\ 0 & 0 & 0 & & \cdots & \sigma+1 \end{bmatrix} \begin{bmatrix} u_{j,0} \\ u_{j,1} \\ u_{j,2} \\ \vdots \\ u_{j,n} \end{bmatrix}$$

3. Download the Mathematica file for the transport equation from the course website. Complete the specification of matrix A in the code. Run the code for wave speeds $c = 0.5$, $c = -0.5$, $c = -1$, $c = -1.5$, and other values of your choice. What do you observe? How can you explain your observations?

observe stability for wave speeds

Courant-Friedrich-Lewy (CFL) criterion $\rightarrow$ $-1 \le c \le 0$

and instability otherwise



4. Perform a stability analysis:

$u_{j+1,m} = (\sigma+1)\, u_{j,m} - \sigma\, u_{j,m+1}$

(a) Let $u_{j,m} = e^{ikx_m}$ and find λ such that $u_{j+1,m} = \lambda u_{j,m}$.

$$u_{j+1,m} = (\sigma+1)\, e^{ikx_m} - \sigma e^{ik(x_m+\Delta x)}$$

$$= e^{ikx_m}\left(\sigma+1-\sigma e^{ik\Delta x}\right)$$

$x_{m+1} = x_m + \Delta x$

$$u_{j+1,m} = u_{j,m}\underbrace{\left(\sigma+1-\sigma e^{ik\Delta x}\right)}_{\lambda}$$

$e^{ik\Delta x} = \cos(k\Delta x) + i\sin(k\Delta x)$

(b) Recalling that $|a + bi|^2 = a^2 + b^2$ for real numbers a and b, show that

$$|\lambda|^2 = 1 + 2\sigma(1 + \sigma)(1 - \cos(k\Delta x)).$$

$$|\lambda|^2 = \left|\sigma+1-\sigma\cos(k\Delta x) - \sigma i \sin(k\Delta x)\right|^2 = \left((\sigma+1)-\sigma\cos(k\Delta x)\right)^2 + \sigma^2\sin^2(k\Delta x)$$

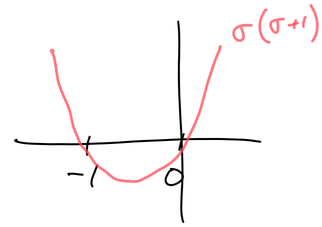
$$= (\sigma+1)^2 - 2\sigma(\sigma+1)\cos(k\Delta x) + \underbrace{\sigma^2\cos^2(k\Delta x) + \sigma^2\sin^2(k\Delta x)}$$

$$= \sigma^2+2\sigma+1 - 2\sigma(\sigma+1)\cos(k\Delta x) + \sigma^2$$

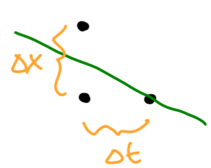
$$|\lambda|^2 = 1 + \underbrace{2\sigma(\sigma+1)}\,\underbrace{\left(1-\cos(k\Delta x)\right)}_{\ge 0}$$

(c) Conclude that $|\lambda| \le 1$ if and only if $-\frac{\Delta x}{\Delta t} \le c \le 0$.

negative or zero if $\sigma(\sigma+1) \le 0$



$-1 \le \sigma \le 0$



$-1 \le \dfrac{c\,\Delta t}{\Delta x} \le 0$

$$\boxed{-\frac{\Delta x}{\Delta t} \le c \le 0}$$

5. Now use a backward difference approximation for $\frac{\partial u}{\partial x}$ and a forward difference approximation for $\frac{\partial u}{\partial t}$. For what wave speeds is this approximation scheme stable?

backwards difference: $\quad \frac{\partial u}{\partial x} \approx \frac{u_{j,m} - u_{j,m-1}}{\Delta x}$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \implies \frac{u_{j+1,m} - u_{j,m}}{\Delta t} + c\,\frac{u_{j,m} - u_{j,m-1}}{\Delta x} = 0$$

$$u_{j+1,m} = u_{j,m} - \frac{c\cdot\Delta t}{\Delta x}\left(u_{j,m} - u_{j,m-1}\right)$$

Let $\sigma = \frac{c\cdot\Delta t}{\Delta x}$. Then: $\boxed{u_{j+1,m} = (1-\sigma)\,u_{j,m} + \sigma\cdot u_{j,m-1}}$

This scheme is stable for $0 \le \sigma \le 1$.

6. Would a centered difference approximation for $\frac{\partial u}{\partial x}$ produce an approximation scheme that works for both positive and negative wave speeds? Try it and find out!

Centered difference: $\quad \frac{\partial u}{\partial x} = \frac{u_{j,m+1} - u_{j,m-1}}{2\Delta x}$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \implies \frac{u_{j+1,m} - u_{j,m}}{\Delta t} + c\,\frac{u_{j,m+1} - u_{j,m-1}}{2\Delta x} = 0$$

$$u_{j+1,m} = u_{j,m} - \frac{c\Delta t}{2\Delta x}\left(u_{j,m+1} - u_{j,m-1}\right)$$

Let $\sigma = \frac{c\cdot\Delta t}{\Delta x}$. Then: $u_{j+1,m} = \frac{\sigma}{2}u_{j,m-1} + u_{j,m} - \frac{\sigma}{2}u_{j,m+1}$.

This scheme turns out to be unstable for all wave speeds.