Recall: Green's Functions

Given a linear differential operator L,

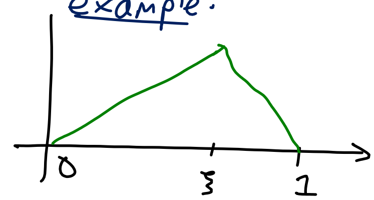
consider $L[u(\vec{x})] = f(\vec{x})$

Green's function G satisfies $L[G(\vec{x})] = \delta_{\xi}(\vec{x})$

Properties:

- G solves the homogeneous diff. eq. $L[G]=0$ at all points $x \neq \xi$
- G satisfies homogeneous boundary conditions
- G is a continuous function of x
- For fixed ξ, the derivative $\frac{\partial G}{\partial x}$ is piecewise C^1, with a single jump discontinuity at $x = \xi$

example:



General solution to $L[u] = f(x)$ is given by

$$u(x) = \int_{\mathbb{R}} G(x;\xi)\, f(\xi)\, d\xi$$

Today: for $(\xi, \eta) \in \mathbb{R}^2$, let $\delta_{\xi,\eta}(x,y)$ be the delta function representing a unit impulse at $(x,y) = (\xi, \eta)$.

$$\delta_{\xi,\eta}(x,y) = 0 \text{ if } (x,y) \neq (\xi,\eta) \quad \text{and} \quad \iint_{\mathbb{R}^2} \delta_{\xi,\eta}(x,y)\, dy\, dx = 1$$

Green's Functions for the 2D Poisson Equation

Math 330

We will examine the Poisson equation

$$-\Delta u = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x, y),$$

which models equilibrium phenomena (such as electrostatic or gravitational potential).

First, recall a few facts from multivariable calculus:

- The **gradient** of $u(x, y)$ is a vector of partial derivatives: $\nabla u = \begin{bmatrix} \partial u / \partial x \\ \partial u / \partial y \end{bmatrix}$.
- The **divergence** of a vector field $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is: $\quad \operatorname{div} \mathbf{v} = \dfrac{\partial v_1}{\partial x} + \dfrac{\partial v_2}{\partial y}$.
- The **divergence theorem** says

$$\iint_\Omega \operatorname{div} \mathbf{F} \, dA = \oint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \, ds$$

  where $\mathbf{F}$ is a vector field, Ω is a region with boundary $\partial\Omega$, and $\mathbf{n}$ is the outward pointing unit normal vector at each point of $\partial\Omega$.

1. Let $f(x, y) = \delta_{\xi,\eta}$ be the 2D delta function at $(\xi, \eta) \in \mathbb{R}^2$, and let $G(x, y; \xi, \eta)$ solve the Poisson equation for this f. Explain why $-\Delta G = 0$ for all $(x, y) \neq (\xi, \eta)$.

We are solving $-\Delta G(x,y;\xi,\eta) = \delta_{\xi,\eta}(x,y) = \begin{cases} 0 & \text{if } (x,y) \neq (\xi,\eta) \\ ? & \text{if } (x,y) = (\xi,\eta) \end{cases}$

2. Explain why $G(x, y; \xi, \eta)$ should really be a function of r alone, where $r = \sqrt{(x - \xi)^2 + (y - \eta)^2}$.

$r = |(x,y) - (\xi,\eta)|$

The Poisson eq. models uniform medium,
and the effect of a unit impulse
depends only on distance, not direction

3. In this case, we seek a radially-symmetric solution to the 2D Laplace Equation. In polar coordinates, the Laplace equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

$x = r\cos\theta$
$y = r\sin\theta$

We want a solution $u(r, \theta)$ that in fact depends only on r.

(a) Simplify the PDE above in the case that $u(r, \theta) = u(r)$.

$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} = 0$ or $u''(r) + \frac{1}{r} u'(r) = 0$

or $r \cdot u''(r) + u'(r) = 0$

(b) Find the general solution to $ru''(r) + u'(r) = 0$. *Hint*: let $v(r) = u'(r)$.

$$r \cdot \frac{dv}{dr} + v = 0$$

$$\frac{dv}{dr} = \frac{-v}{r}$$

$$\int \frac{dv}{v} = \int -\frac{dr}{r}$$

$$e^{\ln|v|} = e^{-\ln|r| + C} = b e^{\ln \frac{1}{|r|}}$$

$$v = b \cdot \frac{1}{r}$$

So: $u'(r) = v(r) = \frac{b}{r}$

$$u(r) = \int \frac{b}{r}\, dr$$

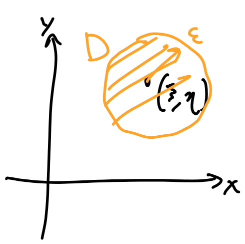
$$\boxed{u(r) = b \cdot \ln(r) + a}$$

4. We now have $G(x, y; \xi, \eta) = a + b\ln(r)$, where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$, and we need $-\Delta G = \delta_{\xi,\eta}$. Why can we choose $a = 0$?

Differentiating G makes the constant not important, so choose $a=0$.

Choose a "baseline potential" of zero.

5. Let D be a disk of radius $\epsilon > 0$ centered at (ξ, η), and let $C = \partial D$. Integrate $-\Delta G = \delta_{\xi,\eta}$ over D to solve for b.



we need: $-\Delta G = \delta_{\xi,\eta}$

integrate:

$$1 = \iint_D \delta_{\xi,\eta}(x,y)\, dy\, dx$$

$$= \iint_D -\Delta G\, dy\, dx$$

from #4

$$= \iint_D -\Delta b \cdot \ln(r)\, dy\, dx$$

$\Delta G = \mathrm{div}(\nabla G)$

$$= -b \iint_D \mathrm{div}\left(\nabla \ln(r)\right) dy\, dx$$

$$= -b \oint_{\partial D} \nabla \ln(r) \cdot \vec{n}\, ds$$

$$= -b \oint_{\partial D} \frac{d}{dr} \ln(r)\, ds$$

$$= -b \oint_{\partial D} \frac{1}{r}\, ds$$

$$= -b\left(\frac{1}{\varepsilon} \cdot 2\pi\varepsilon\right)$$

$$1 = -2\pi b$$

Thus, $b = \frac{-1}{2\pi}$.

6. Write the Green's function for the 2D Poisson equation.

$$G(x, y; \xi, \eta) = \frac{-1}{2\pi} \ln(r) = \frac{-1}{2\pi} \ln\left[(x-\xi)^2 + (y-\eta)^2\right]^{1/2}$$

$$= \frac{-1}{4\pi} \ln\left[(x-\xi)^2 + (y-\eta)^2\right]$$