Green's Functions for the 2D Poisson Equation

Math 330

We will examine the Poisson equation

$$-\Delta u = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = f(x,y),$$

which models equilibrium phenomena (such as electrostatic or gravitational potential).

First, recall a few facts from multivariable calculus:

- The **gradient** of $u(x,y)$ is a vector of partial derivatives: $\nabla u = \begin{bmatrix} \partial u/\partial x \\ \partial u/\partial y \end{bmatrix}$.
- The **divergence** of a vector field $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is: $\quad \operatorname{div} \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y}$.
- The **divergence theorem** says

$$\iint_\Omega \operatorname{div} \mathbf{F} \, dA = \oint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} \, ds$$

  where $\mathbf{F}$ is a vector field, Ω is a region with boundary $\partial\Omega$, and $\mathbf{n}$ is the outward pointing unit normal vector at each point of $\partial\Omega$.

1. Let $f(x,y) = \delta_{\xi,\eta}$ be the 2D delta function at $(\xi,\eta) \in \mathbb{R}^2$, and let $G_0(x,y;\xi,\eta)$ solve the Poisson equation for this f. Explain why $-\Delta G = 0$ for all $(x,y) \neq (\xi,\eta)$.

2. Explain why $G(x,y;\xi,\eta)$ should really be a function of r alone, where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$.

3. In this case, we seek a radially-symmetric solution to the 2D Laplace Equation. In polar coordinates, the Laplace equation becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

We want a solution $u(r,\theta)$ that in fact depends only on r.

(a) Simplify the PDE above in the case that $u(r,\theta) = u(r)$.

(b) Find the general solution to $ru''(r) + u'(r) = 0$. *Hint*: let $v(r) = u'(r)$.

4. We now have $G(x, y; \xi, \eta) = a + b\ln(r)$, where $r = \sqrt{(x-\xi)^2 + (y-\eta)^2}$, and we need $-\Delta G = \delta_{\xi,\eta}$. Why can we choose $a = 0$?

5. Let D be a disk of radius $\epsilon > 0$ centered at (ξ, η), and let $C = \partial D$. Integrate $-\Delta G = \delta_{\xi,\eta}$ over D to solve for b.

6. Write the Green's function for the 2D Poisson equation.