2 D Heat Equation on a disk

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \qquad \text{solution } u(t,x,y)$$

$\Downarrow$ polar coordinates $(x,y) \to (r,\theta)$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

Seek product solutions $u(t,r,\theta) = G(t)\, f(r)\, q(\theta)$

find: $G(t) = e^{-\lambda t}$

$q(\theta) = \cos(m\theta)$ or $\sin(m\theta)$

$f(r)$ satisfies $r^2 f''(r) + r f'(r) + \lambda r^2 f(r) = \mu f(r)$

$$r^2 f''(r) + r f'(r) + (\lambda r^2 - \mu) f(r) = 0$$

$\frac{df}{dr} = \frac{df}{dz} \cdot \frac{dz}{dr}$

$\frac{df}{dr} = \frac{df}{dz} \cdot \sqrt{\lambda}$

rescale: let $z = \sqrt{\lambda}\, r$

$m^2 = \mu$

$$\underbrace{z^2 f''(z) + z f'(z) + (z^2 - m^2) f(z) = 0}$$

Bessel's equation of order $m \geq 0$

Bessel's Equation

Math 330

The following ODE is is known as **Bessel's differential equation of order** $m > 0$:

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - m^2) f = 0$$

If m is not an integer, then Bessel's equation of order m has two linearly independent solutions denoted $J_m(z)$ and $J_{-m}(z)$, with

$$\boxed{J_m(z) = \sum_{k=0}^{\infty} \frac{(-1)^k z^{m+2k}}{2^{m+2k} k!\, \Gamma(m+k+1)},}$$

where $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$ is the Gamma function. The function $J_m(z)$ is the **Bessel function of the first kind of order** m.

If m is an integer, then $\boxed{J_{-m}(z) = (-1)^m J_m(z),}$ so these two solutions are not linearly independent. A second solution linearly independent to $J_m(z)$ is given by

$$\boxed{Y_m(z) = \lim_{\alpha \to m} \frac{J_\alpha(z)\cos(\alpha\pi) - J_{-\alpha}(z)}{\sin(\alpha\pi)}}$$

the **Bessel function of the second kind of order** m.

1. Plot the functions $J_m(z)$ and $Y_m(z)$ for various m to see the shape of their graphs. What do you observe about these functions?

2. Suppose we impose the boundary conditions $|f(0)| < \infty$ and $f(a) = 0$. The condition at $r = 0$ is known as a *singular* boundary condition, and the other boundary condition is imposed at $r = a$. Which of the Bessel functions can satisfy these boundary conditions? What eigenvalues do these boundary conditions determine?

3. The Bessel functions of the first kind satisfy the following orthogonality property:

$$\int_0^1 J_m\left(\sqrt{\lambda_{mp}}\, r\right) J_m\left(\sqrt{\lambda_{mq}}\, r\right) r\, dr = 0, \qquad p \neq q,$$

where $\sqrt{\lambda_{mn}} = z_{mn}$ denotes the nth zero of $J_m(z)$. Use Mathematica to confirm this for some specific m, p, and q. (The zeros of J_m are transcendental numbers, but Mathematica's function BesselJZero returns approximations of them.)

Heat equation on a disk:

root of $J_m(z)$

Solutions:

$$u(t,r,\theta) = e^{-z_{mn}^2 t} J_m(z_{mn} r) \sin(m\theta)$$

$$\hat{u}(t,r,\theta) = e^{-z_{mn}^2 t} J_m(z_{mn} r) \cos(m\theta)$$

General Solution:

$$u(t,r,\theta) = \frac{1}{2} \sum_{n=1}^{\infty} a_{0,n} u_{0,n}(t,r) + \sum_{m,n=1}^{\infty} \left[a_{m,n} u_{m,n}(t,r,\theta) + b_{m,n} \hat{u}_{m,n}(t,r,\theta) \right]$$

$$u_{0,n}(t,r) = e^{-z_{0,n}^2 t} J_0(z_{0,n} r)$$