Homework 8

Math 330

due at 5pm on Thursday, November 30, 2023

Solve the following problems and communicate your solutions clearly. Explain your work using complete sentences, and include diagrams as appropriate.

For this homework, you must type your solutions in LaTeX. Some of these problems require computations, which you may do in Mathematics or other software. Your LaTeX file should contain a summary of what you computed, including screenshots of plots as appropriate, but you should not reproduce large amounts of code in your LaTeX file. Your LaTeX file should clearly communicate your solutions and observations resulting from your computational work. You may submit your Mathematica notebook (or other code) as a supplement to your LaTeX file, but this is not required.

Plots/graphs may be drawn by hand or using technology and inserted into your LaTeX document. Make sure your solutions are easy to read, in order, and clearly labeled. Upload a single file containing your solutions to the Homework 8 assignment on Moodle.

1. (6 points) Exercise 5.2.1

For part (b), you may use Mathematica code from class. In your LaTeX file, include a few plots of your computational results.

2. (7 points) Exercise 5.2.8, modified: Consider the initial-boundary value problem for the lossy diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \alpha u, \qquad u(t,0) = u(t,1) = 0, \qquad u(0,x) = f(x), \qquad t \geq 0, \quad 0 \leq x \leq 1$$

where $\alpha > 0$ is a positive constant.

(a) Devise an explicit finite difference method for computing a numerical approximation to the solution. Use a forward difference approximation for $\frac{\partial u}{\partial t}$ and a centered difference approximation for $\frac{\partial^2 u}{\partial x^2}$. Your finite difference equation should be in the form:

$$u_{j+1,m} = \quad \text{[some terms involving time index } j\text{]}$$

(b) Perform a stability analysis of your finite difference equation. That is, let $u_{j,m} = e^{ikx_m}\lambda^j$ and show that

$$\lambda = 1 - 4\mu \sin^2\left(\frac{1}{2}k\,\Delta x\right) - \alpha\,\Delta t.$$

What condition on Δx and Δt guarantees that $|\lambda| \leq 1$, and thus that the approximation scheme is stable?

3. (7 points) Exercise 5.4.1

You may use Mathematica code from class. In your LaTeX file, include plots of your computational results.

4. (10 points) Consider the following population dispersion model with a growth term. (This model assumes that the population has logistic growth and the spread of the population can be modeled by diffusion.)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u(1-u), \qquad u(t,0) = u(t,1) = 0, \qquad u(0,x) = \sin(\pi x), \qquad t \geq 0, \quad 0 \leq x \leq 1$$

(a) Write the partial difference equation for the PDE using a forward difference approximation for $\frac{\partial u}{\partial t}$ and a centered difference approximation for $\frac{\partial^2 u}{\partial x^2}$.

(b) Implement your approximation scheme (use the code from class as a template) and compute an approximate solution to the the PDE above with $\alpha = 1$. Choose Δt small enough so that you get a stable solution. What happens as $t \to \infty$?

(c) Now let $\alpha = 20$. What happens as $t \to \infty$? What if you use the initial condition $u(0, x) = 0.1 \sin(\pi x)$?

(d) For the long-term behavior obtained in parts (b) and (c), explain in two sentences or less why your answers seem reasonable (relate your solutions to the physical context).