

TOPOLOGY DAY 12

22 October 2026

DEFINITION: If S and T are topological spaces, the **disjoint union** $S \amalg T$ consists of all the points in S together with all of the points in T , with all of these points regarded as distinct. Any points in $S \cap T$ occur twice in $S \amalg T$.
A subset $Q \subset S \amalg T$ is open if $Q \cap S$ is open in S and $Q \cap T$ is open in T .

EXAMPLE: Let $S = \{x, y, z\}$ and $T = \{w, x, y\}$.

To write elements in $S \amalg T$, we must keep track of which elements come from S , and which come from T . We can use subscripts for this, writing

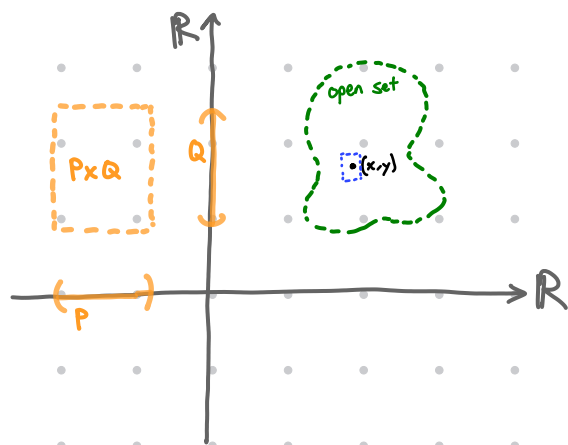
$$S \amalg T = \{x_s, y_s, z_s, w_t, x_t, y_t\}.$$

Now suppose that S has the discrete topology and T has the indiscrete topology. Then an open set in $S \amalg T$ may contain any elements of S and either all or no elements of T .

NOTE: If S and T are finite sets, say S has m elements and T has n elements, then $S \amalg T$ has $m+n$ elements.

DEFINITION: If S and T are two sets, the **Cartesian product** $S \times T$ consists of all pairs (s, t) where $s \in S$ and $t \in T$. If S and T are topological spaces, the **product topology** is the topology which has a basis consisting of all products $P \times Q$ of an open subset $P \subset S$ with an open subset $Q \subset T$.

EXAMPLE: $\mathbb{R} \times \mathbb{R} \cong \mathbb{R}^2$



Disjoint Unions

MATH 348

1. Describe each of the following disjoint unions. What are the points of each space? What sets are open in each space?

(a) $\mathbb{R} \amalg \mathbb{R}$



$$\mathbb{R} \amalg \mathbb{R} = \{(x, i) \mid x \in \mathbb{R}, i \in \{1, 2\}\}$$

If A open in $\mathbb{R}$,

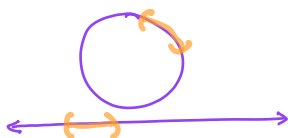
and B is open in $\mathbb{R}$,

then $A \amalg B$ is open in $\mathbb{R} \amalg \mathbb{R}$.

(b) $\{-1\} \amalg \{+1\} = \{-1, +1\} = S^0$

with the discrete topology

(c) $\mathbb{R} \amalg S^1$



2. Let S and T be compact topological spaces. Prove that $S \amalg T$ is compact.

Let S and T be compact topological spaces. Let $\mathcal{C}$ be an open cover of $S \amalg T$. Then $\mathcal{C}_S = \{U \cap S \mid U \in \mathcal{C}\}$ is an open cover of S , so it has a finite refinement $\overline{\mathcal{C}}_S$. Similarly, $\mathcal{C}_T = \{U \cap T \mid U \in \mathcal{C}\}$ is an open cover of T , so it has a finite refinement $\overline{\mathcal{C}}_T$.

Then $\{U \in \mathcal{C} \mid U \cap S \in \overline{\mathcal{C}}_S \text{ or } U \cap T \in \overline{\mathcal{C}}_T\}$ is a finite refinement of $\mathcal{C}$, covering $S \amalg T$. So $S \amalg T$ is compact.

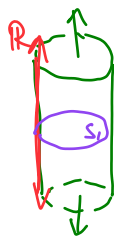
3. Under what circumstances is $S \amalg T$ a Hausdorff space? Formulate and prove a statement similar to #2 above.

Product Spaces

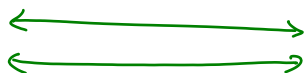
MATH 348

- Describe each of the following product spaces. What are the points of each space? What sets are open in each space?

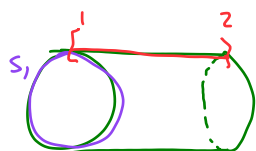
(a) $\mathbb{R} \times S^1$ infinite cylinder



(b) $\mathbb{R} \times S^0 = \mathbb{R} \sqcup \mathbb{R}$



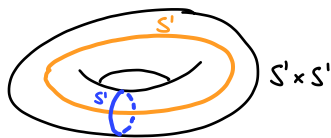
(c) $S^1 \times [1, 2]$



hollow cylinder
finite length

(d) $S^1 \times S^1$

torus



- For any product space $S \times T$, show that the projection maps $\pi_1 : S \times T \rightarrow S$ and $\pi_2 : S \times T \rightarrow T$ are continuous.

We will come back
to this next time.

- Let S be a connected topological space, and let T be another connected topological space. Prove that $S \times T$ is connected.