Three's A Crowd: An Exploration of Subprime Tribonacci Sequences

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Abstract

A subprime Fibonacci sequence follows the Fibonacci recurrence, where the next term in a sequence is the sum of the two previous terms, except that composite sums are divided by their least prime factor. We extend the recurrence to three terms, investigating subprime *tribonacci* sequences. It appears that all such sequences eventually enter a repeating cycle. We compute cycles arising from more than one billion sequences, classifying them as *trivial*, *tame*, and *wild*. We further investigate questions of parity and primality in subprime tribonacci sequences. In particular, we show that any nonzero subprime tribonacci sequence eventually contains an odd term.

1 Introduction

Suppose we compute terms of the Fibonacci sequence, but whenever we encounter a composite number, we divide it by its least prime factor before writing it in the sequence. We obtain a sequence of integers:

$$0, 1, 1, 2, 3, 5, 4, 3, 7, 5, 6, \ldots$$

In other words, when computing the usual Fibonacci sequence, we add $3+5$ to obtain 8, but in this sequence, we divide 8 by 2 and obtain 4 instead. We then encounter $5+4=9$, but we divide by 3, obtaining 3 as the next term in the sequence. The sequence we compute in this fashion has become known as *Conway's subprime Fibonacci sequence* [5, 6].

More generally, starting with any two nonnegative integers a_0 and a_1, we obtain a subprime Fibonacci sequence by recursively applying the rule

$$a_n = \frac{a_{n-1} + a_{n-2}}{\operatorname{lpf}(a_{n-1} + a_{n-2})},$$

where $\operatorname{lpf}(n)$ denotes the least prime factor of a composite number n, or 1 if n is not composite. For example, $\operatorname{lpf}(8) = 2$, $\operatorname{lpf}(45) = 3$, and $\operatorname{lpf}(5) = 1$. Guy, Khovanova, and Salazar studied subprime Fibonacci sequences with starting terms up to 10^5 and observed that every such sequence either becomes constant or enters one of six different cycles [4].

What if, instead of a recurrence relation involving two previous terms, we use a recurrence of three terms? Would interesting patterns emerge? Specifically, we begin with any three nonnegative integers a_0, a_1, a_2 and recursively apply the rule

$$a_n = \frac{a_{n-1} + a_{n-2} + a_{n-3}}{\operatorname{lpf}(a_{n-1} + a_{n-2} + a_{n-3})}. \tag{1}$$

We call the resulting sequences *subprime tribonacci sequences*.

For example, if we start with the integers $107, 29, 23$, then we obtain the sequence

$$107, 29, 23, \underline{53, 35, 37}, 25, 97, \underline{53, 35, 37}, \ldots$$

As soon as we see the terms $53, 35, 37$ appear consecutively a second time in the sequence, we realize that we have entered a cycle: the sequence will forever repeat the five terms $53, 35, 37, 25, 97$.

Formally, the *length* of a cycle is the smallest positive integer k such that $a_n = a_{n+k}$ for all n greater than some integer N. The first N terms of the sequence need not be part of the cycle. We specify that k is *smallest* so that the length is well-defined, rather than allowing lengths of multiple iterations of the cycle. Furthermore, we sometimes refer to a cycle of length k as a *k-cycle*.

Thus, the previous subprime tribonacci sequence exhibits a 5-cycle, which is already interesting, since it is half the length of the smallest nontrivial cycle known to occur in subprime Fibonacci sequences [4].

We are aware of only one reference to subprime tribonacci sequences in the mathematics literature. Mihai Caragiu mentions these sequences in *Sequential Experiments with Primes* [1, Section 4.3], identifying four cycles that result as long-term behavior of these sequences.

In this paper, we explore the cycles that appear in subprime tribonacci sequences. Are there really only four such cycles, or possibly more? Do these cycles exhibit any interesting properties, such as patterns in parity or primality? Is there perhaps a cycle consisting entirely of even numbers? Our search for answers takes us through the realms of both computation and proof, even touching on some advanced topics in algebra and number theory.

2 Does it repeat? Does it repeat?

Subprime tribonacci sequences are defined by a third-order recurrence (Equation (1)), meaning that each term is computed from the three previous terms. Accordingly, a subprime tribonacci sequence enters a cycle whenever a triple of consecutive terms appears more than once in the sequence; the sequence must then repeat the cycle indefinitely. A subprime tribonacci sequence that does not enter a cycle must not be bounded, for there exist only finitely many triples of nonnegative integers below any bound. Therefore, any subprime tribonacci sequence must either enter a cycle or contain a subsequence that increases without bound.

We computed subprime tribonacci sequences from more than one billion starting triples and found that each sequence enters a cycle eventually. Furthermore, we observe three distinct categories of cycles that appear. *Trivial cycles* consist of a single number that repeats. Other cycles are non-constant but consist of numbers that share a common factor; we call these *tame cycles*. We show that we have found all possible tame cycles of length 2 or 4, though other lengths may be possible. All other cycles we call *wild cycles*; our investigation revealed nine wild cycles, with lengths up to 3174.

2.1 Trivial Cycles

Some subprime tribonacci sequences converge to a limit, a constant which repeats indefinitely. Such a limit is a cycle of length 1, which we call a *trivial cycle*. The following sequence not only exhibits a trivial cycle, but also shows that a sequence leading to a trivial cycle may be initially non-constant:

$$7, 0, 21, 14, 7, 21, 21, 7, 7, 7, \ldots$$

If a subprime tribonacci sequence reaches a limit a, then Equation (1) implies that

$$a = \frac{3a}{\operatorname{lpf}(3a)}.$$

This means that either $a = 0$ or a is an odd integer greater than 1.

In the example sequence above, it is no accident that all of the nonconstant terms are multiples of the limit 7. A trivial cycle consisting of the odd integer a can only arise from three starting integers that have a common factor of a. This is a consequence of the following proposition, which shows that the greatest common divisor (gcd) of three consecutive terms must either stay the same or decrease by a prime factor with each new term of the subprime tribonacci sequence.

Proposition 1. *Let a, b, c, d be four consecutive terms in a subprime tribonacci sequence. Then*

$$\gcd(a, b, c) = q \cdot \gcd(b, c, d),$$

where either $q = 1$ or $q = \operatorname{lpf}(a + b + c)$.

Proof. We consider two cases. In the first case, suppose $a+b+c$ is prime. Then $d = a+b+c$ and

$$\gcd(a, b, c) = \gcd(b, c, a + b + c) = \gcd(b, c, d),$$

and the conclusion holds with $q = 1$.

In the second case, suppose $a + b + c$ is composite. Then let $p = \operatorname{lpf}(a + b + c) > 1$, and thus $d = \frac{a+b+c}{p}$. Let $g = \gcd(a, b, c) = \gcd(b, c, a + b + c)$. If $a + b + c$ is divisible by gp, then $\gcd\left(b, c, \frac{a+b+c}{p}\right) = g$, and the statement holds with $q = 1$. Otherwise, $a + b + c$ is not divisible by gp, which means that g does not divide $\frac{a+b+c}{p}$. Since g divides $a + b + c$, this implies that g has a factor of p. Thus, $\gcd\left(b, c, \frac{a+b+c}{p}\right) = \frac{g}{p}$, so the conclusion holds with $q = p$. □

Having described all trivial cycles, we turn to consider cycles that are not constant.

2.2 Tame Cycles

Consider the following subprime tribonacci sequences:

$$\begin{gathered} 7, 14, 7, 14, 7, 14, 7, 14, \ldots \\ 35, 35, 70, 70, 35, 35, 70, 70, \ldots \\ 11, 44, 33, 44, 11, 44, 33, 44, \ldots \\ 17, 68, 68, 51, 17, 68, 68, 51, \ldots \end{gathered}$$

These sequences exhibit non-constant cycles with the property that all terms share a common prime factor. We call such cycles *tame cycles*. Our computational investigation reveals tame cycles of lengths 2 and 4, which we describe explicitly below.

There are no cycles of length 3, for if a, b, c form a cycle of length 3, then each term must equal $\frac{a+b+c}{\operatorname{lpf}(a+b+c)}$ and the cycle is trivial (length 1). It is an open question whether there exist tame cycles of length greater than 4.

All subprime tribonacci cycles of length 2 are of the form $a, 2a, a, 2a, \ldots$, where a has no prime factor smaller than 5. Clearly the terms of such cycles alternate even and odd. The restriction on factors of a ensures that the denominators in Equation (1) alternate between 2 and 5, which is essential, as shown in the proof of the following theorem.

Theorem 2. *The only cycles of length 2 alternate between integers a and $2a$, where a is an odd integer with no prime factor smaller than 5.*

Proof. Suppose that integers a and b form a cycle of length 2. We examine possible parities of a and b.

First, suppose that both a and b are even. Then Equation (1) implies that $b = \frac{2a+b}{2}$, and so $a = \frac{b}{2}$. Similarly, $a = \frac{2b+a}{2}$, which yields $b = \frac{a}{2}$. The only solution is $a = b = 0$, which is a trivial cycle, not a cycle of length 2..

Next, suppose that both a and b are odd. Then Equation (1) gives

$$b = \frac{2a + b}{\operatorname{lpf}(2a + b)} \qquad \text{and} \qquad a = \frac{2b + a}{\operatorname{lpf}(2b + a)}.$$

Let $p = \text{lpf}(2a+b)$ and $q = \text{lpf}(2b+a)$. Since a and b are odd, p and q must be odd as well. We rewrite the equations above as

$$\frac{b(p-1)}{2} = a \qquad \text{and} \qquad \frac{a(q-1)}{2} = b.$$

Neither p nor q may equal 1, for this would produce $a = b = 0$ and a trivial cycle. However, if $p > 3$, then $\frac{p-1}{2} > 1$, and so $b > a$. Similarly, if $q > 3$, then $\frac{q-1}{2}$, and so $a > b$. Thus, p and q may not *both* be greater than 3. The only remaining option is that at least one of p or q must equal 3, but this again implies that $a = b$ and we have only a trivial cycle.

Lastly, suppose that a and b have different parities. Without loss of generality, let a be odd and b even. Then $2a + b$ is even, so Equation (1) yields $b = \frac{2a+b}{2}$. This implies that $a = \frac{b}{2}$, and thus $b = 2a$. Returning to Equation (1), we find

$$a = \frac{2b+a}{\text{lpf}(2b+a)} = \frac{5a}{\text{lpf}(5a)},$$

which can only hold if $\text{lpf}(5a) = 5$. Thus, we obtain a cycle of the form $a, 2a$ whenever a is an odd integer with no prime factor smaller than 5. $\square$

Having classified all cycles of length 2, we now turn to cycles of length 4. Intriguingly, we classify all tame cycles of length 4, though we cannot rule out the existence of a 4-cycle whose terms have no common prime factor. Among tame 4-cycles, we find two distinct patterns. Some tame 4-cycles have the form $a, a, 2a, 2a$, where a has no prime factor smaller than 5. Other tame 4-cycles consist of some permutation of $b, 3b, 4b, 4b$, for an integer b with no prime factor smaller than 11. Our next theorem shows that all tame 4-cycles fit one of these patterns.

Theorem 3. *All tame cycles of length 4 fit one of the following patterns:*

- *The cycle repeats the four terms $a, a, 2a, 2a$, where a has no prime factor smaller than 5.*
- *The cycle repeats some permutation of the four terms $b, 3b, 4b, 4b$, where b has no prime factor smaller than 11.*

Proof. Suppose there is a tame cycle of length 4, consisting of the integers a, b, c, d in that order. From Equation (1), these numbers must satisfy:

$$a = \frac{b+c+d}{\text{lpf}(b+c+d)}, \qquad b = \frac{a+c+d}{\text{lpf}(a+c+d)}, \qquad c = \frac{a+b+d}{\text{lpf}(a+b+d)}, \qquad d = \frac{a+b+c}{\text{lpf}(a+b+c)}. \tag{2}$$

We consider the parities of a, b, c, d in five cases.

Case 1. Suppose that all of a, b, c, d are even. In this case, each of the denominators in Equation (2) is 2. This implies that each of a, b, c, d is greater than the average of the other three numbers, which is impossible.

Case 2. Suppose that one of a, b, c, d is odd, and the rest are even. Without loss of generality, let a be odd and b, c, d be even. Then $a + b + c$ is odd, which implies d is odd—a contradiction.

Case 3. Suppose that two of a, b, c, d are odd. This is the most involved case. Without loss of generality, either a and b are odd, or a and c are odd. We address each of these two subcases separately.

For the first subcase, let a and b be odd (thus c and d are even). This implies that $\text{lpf}(a+b+c) = \text{lpf}(a+b+d) = 2$. Together with Equation (2), we have $2d = a+b+c$ and $2c = a+b+d$, which implies $c = d = a+b$.

Let $q = \text{lpf}(b+c+d)$ and $r = \text{lpf}(a+c+d)$. Then Equation (2) implies that $a\frac{q-2}{3} = b$ and $b\frac{r-2}{3} = a$. Since a and b are positive, it must be that $q, r > 2$. We examine several possibilities:

- If $q = 3$, then $a = 3b$ and $r = 11$. This implies $c = d = 4b$, and we have a cycle of the form $3b, b, 4b, 4b$. Since $r = \operatorname{lpf}(a + c + d) = \operatorname{lpf}(11b) = 11$, we see that b must satisfy $\operatorname{lpf}(b) \geq 11$.
- Similarly, if $r = 3$, then $q = 11$. This yields a cycle of the form $a, 3a, 4a, 4a$ for any integer a such that $\operatorname{lpf}(a) \geq 11$.
- If $q = 5$, then $a = b$ and $r = 5$ also. We obtain a cycle of the form $a, a, 2a, 2a$ for any integer a such that $\operatorname{lpf}(a) \geq 5$.
- If both q and r are greater than 5, then $\frac{q-2}{3} > 1$ and $\frac{r-2}{3} > 1$. This implies that $b > a$ and also $a > b$, which is impossible.

For the second subcase, let a and c be odd (thus b and d are even). We then have that $\operatorname{lpf}(a + b + c) = \operatorname{lpf}(a + c + d) = 2$. Together with Equation (2), this implies that $2d = a + b + c$ and $2b = c + d + a$, so $b = d = a + c$.

Let $q = \operatorname{lpf}(b + c + d)$ and $s = \operatorname{lpf}(a + b + d)$. Now Equation (2) implies that $a\frac{q-2}{3} = c$ and $c\frac{s-2}{3} = a$. Since a and c are positive, we have $q, s, > 2$. We again examine several possibilities, which are analogous to those previously considered:

- If $q = 3$, then $a = 3c$ and $s = 11$. This yields a cycle of the form $3c, 4c, c, 4c$ for $\operatorname{lpf}(c) \geq 11$.
- If $s = 3$, then $c = 3a$ and $q = 11$, which yields a cycle of the form $a, 4a, 3a, 4a$ for $\operatorname{lpf}(a) \geq 11$.
- If $q = 5$, then $a = c$ and $s = 5$. We obtain the sequence $a, 2a, a, 2a$, which is a cycle of length 2, not of length 4.
- If both $q, s, > 5$, then $\frac{q-2}{3} > 1$ and $\frac{s-2}{3} > 1$, which implies that $c > a$ and $a > c$, an impossibility.

Case 4. Suppose that one of a, b, c, d are even, and the rest are odd. Without loss of generality, let a, b, and c be odd. Then their sum is odd, and so d is odd—a contradiction.

Case 5. Finally, suppose that all of a, b, c, d are odd. By the definition of a tame cycle, a, b, c, d share a common prime factor, which must be odd. Thus, all of the denominators in Equation (2) are at least 3. This means that each of a, b, c, d is less than or equal to the average of the other three. This is only possible if each number is equal to the average of the other three, which implies that $a = b = c = d$ and we have a cycle of length 1, not of length 4.

We see that only Case 3 results in cycles of length 4, and these cycles fit the two patterns given in the statement of the theorem. □

We emphasize that Theorem 3 does not rule out the possibility that there is a cycle of length 4 consisting of odd integers that *don't* share a common factor. Such a cycle would not be a tame cycle, but rather a wild cycle, which we now discuss.

2.3 Wild Cycles

All cycles consisting of numbers that do not share a common prime factor we call *wild cycles*. This name alludes to the seeming unpredictability that we observe in these cycles, especially in their lengths and the magnitudes of values that they contain.

We computed subprime tribonacci sequences starting with all triples a, b, c of nonnegative integers up to and including 1000. That is, we examined slightly more than one billion sequences (1001^3, to be exact), computing terms for each until we identified an eventual cycle. We found that every sequence that we computed enters a cycle.

Besides the trivial and tame cycles discussed previously, we found nine wild cycles. The shortest wild cycle, which has length 5, already appeared in the Introduction. Seven other cycles have lengths from 6 to 203. The longest cycle we found has length 3174. These wild cycles are summarized in Table 1, which also displays the largest term and the number of prime terms in each cycle.

Intriguingly, all of the wild cycles that we found consist entirely of odd numbers, an observation we will explore further below. We notice that these odd numbers are frequently prime: 2473 of the 3767 total terms (65.6%) in the all of the wild cycles are prime. Among these terms we find 821 unique primes, including *all* primes between 11 and 929. The largest term in any cycle is 454507, which is prime and appears in the 3174-cycle.

Cycle length	Terms	Largest term	Num primes
5	$25, 97, 53, 35, 37$	97	3
6	$37, 139, 73, 83, 59, 43$	139	6
29	$59, 617, 983, \ldots, 209, 251, 307$	10169	21
30	$47, 151, 211, \ldots, 313, 335, 1279$	3251	23
70	$59, 221, 439, \ldots, 313, 1777, 1037$	16063	43
93	$19, 149, 43, \ldots, 79, 83, 47$	9257	63
157	$67, 151, 919, \ldots, 1087, 289, 701$	69653	90
203	$17, 113, 65, , \ldots, 23, 31, 65$	145969	136
3174	$11, 71, 37, \ldots, 17, 31, 29$	454507	2088

Table 1: Summary of known wild cycles, each printed with its smallest term first.

Moreover, a perceptive reader might notice that all of the numbers in the "largest term" column of Table 1 are prime! This is no coincidence, but rather a property of subprime tribonacci cycles consisting of odd terms.

Proposition 4. *If a subprime tribonacci cycle consists entirely of odd numbers, then the largest term in the cycle is prime.*

Proof. Let d be the largest term in the cycle, and let a, b, c be the three terms preceding d in the cycle. Note that $a + b + c$ is odd.

If $a + b + c$ is composite, then $\operatorname{lpf}(a + b + c) \geq 3$, which implies $d = \frac{a+b+c}{\operatorname{lpf}(a+b+c)} \leq \frac{a+b+c}{3}$. Thus d is not greater than the average of a, b, and c. At least one of a, b, and c must be greater than their average, and thus greater than d, which is a contradiction.

Therefore, $a + b + c$ must be prime, so $d = a + b + c$ is prime. $\square$

We invite the reader to consider whether Table 1 list all possible wild cycles. Perhaps further computational investigation will turn up additional cycles. It would be particularly interesting to find a wild cycle that contains one or more even numbers, or to prove that such a cycle does not exist—we invite the reader to ponder this question, or to read the next section.

Table 2 displays the frequency of each cycle that occurs for starting values in various ranges. Fascinatingly, the 3174-cycle is by far the most common cycle, occurring in 93.4% of the 1001^3 subprime tribonacci sequences that we computed. Most sequences result in wild cycles: trivial and tame cycles occur in 3.53% of sequences with starting values not greater than 10, but the percentage of such cycles drops to a mere 0.0078% when considering starting values up to 1000. Cycles of length 70 and 157 do not occur at all for triples of starting values all not greater than 100. Computing sequences in lexicographical order by starting triple, we first find a cycle of length 157 when starting with the values $31, 108, 88$, and the first 70-cycle occurs in the sequence starting with the values $38, 57, 118$.

Cycle length	$a, b, c \leq 10$	$a, b, c \leq 100$	$a, b, c \leq 1000$
1	37	1088	75889
2	3	92	1124
4	7	409	11113
5	5	1437	1169622
6	40	61608	59904236
29	0	19	18887
30	0	8	11559
70	0	0	3551
93	0	1256	1356276
157	0	0	9182
203	0	3600	3424343
3174	1239	960784	937017219

Table 2: Distribution of cycle lengths by starting triple of nonnegative integers a, b, c. Cycles of length 1 are trivial, cycles of lengths 2 and 4 are tame, and all other cycles are wild.

3 Truly Odd Results

Our examination of cycles reveals no even terms among the wild cycles, which seem to occur in the vast majority of subprime tribonacci sequences. Why is this? We now consider what patterns of even and odd terms are possible in subprime tribonacci sequences.

It follows from the subprime tribonacci recurrence, Equation (1), that only certain patterns of even and odd terms are possible in subprime tribonacci sequences. If the sum of three consecutive terms is even, then the least proper factor of this sum is 2, and the next term may be even or odd. However, if the sum of three consecutive terms is odd, then the next term *must* be odd. The directed graph in Figure 1 illustrates these observations. The eight nodes are labeled with the eight possible parity sequences (E for even, O for odd) for three consecutive terms; arrows between nodes indicate the possible parity of the next term.

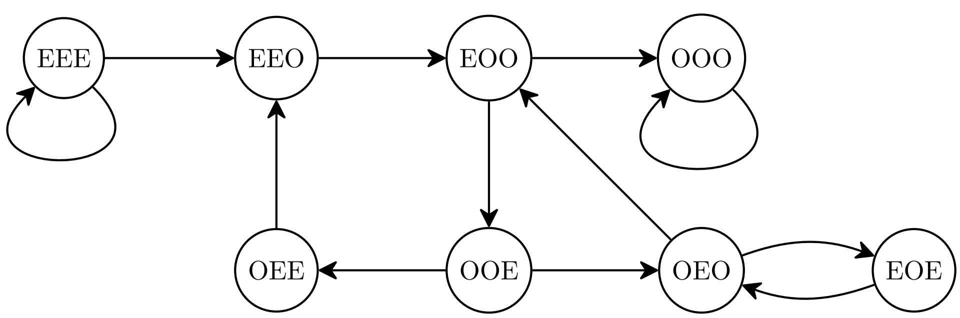

Figure 1: Parity graph for three consecutive terms of a subprime tribonacci sequence.

For example, suppose that a, b, c are consecutive terms in a subprime tribonacci sequence. If a and c are odd, and b is even, then these terms are represented by the node labeled OEO. Since $a + b + c$ is even, the next term is $d = \frac{a+b+c}{2}$, which may be even or odd. Thus, the three terms b, c, d may have parities EOO or EOE. This is indicated by the arrows OEO $\to$ EOO and OEO $\to$ EOE in the graph.

The parity graph contains one *absorbing node*: the node OOO, from which there are no arrows to any other nodes. We thus have the following proposition.

Proposition 5. *If three consecutive terms are odd, then* all *subprime tribonacci terms after these three will also be odd.*

Proof. If a, b, c are three consecutive odd terms, then $a + b + c$ is odd as well. Dividing $a + b + c$ by its least proper factor results in an odd number for the next term in the sequence. The result follows by induction. $\square$

We notice other cycles (i.e., directed paths that return to their starting node) in the parity graph. For example, the two edges connecting nodes OEO and EOE form a cycle in the graph, corresponding to subprime tribonacci cycles that alternate even and odd terms. We have seen this alternating pattern in the tame cycle of length 2, as well as in one pattern for 4-cycles. Another graph cycle appears in the square with nodes EEO, EOO, OOE, and OEE. This corresponds to subprime tribonacci cycles that contain alternating pairs of even and odd terms, which we have seen in the other tame 4-cycles.

The parity graph also contains a triangle formed by the nodes EOO, OOE, and OEO. However, we have seen that there is no subprime tribonacci 3-cycle. Whether there exist subprime tribonacci cycles whose length is a *multiple* of 3 exhibiting the parity pattern EOOEOO... is an open question.

Intriguingly, we note the self-loop at the EEE node of the parity graph. Already curious about the relative lack of even terms in subprime tribonacci sequences, we now realize that there may exist subprime tribonacci sequences consisting *entirely* of even terms—the parity graph does not rule this out, in any case. Perhaps we can construct such a sequence?

Suppose a, b, c are the initial terms of a nonzero subprime tribonacci sequence consisting entirely of even terms. In this sequence, the sum of any three consecutive terms must be even, and thus has least proper factor 2. Applying the recurrence relation from Equation (1), we find that the sequence can then be written as:

$$a, b, c, \frac{a+b+c}{2}, \frac{a+3b+3c}{4}, \frac{3a+5b+9c}{8}, \frac{9a+15b+19c}{16}, \ldots. \tag{3}$$

The numerators in Equation (3) form a *weighted tribonacci sequence*: each numerator is a weighted sum of the three preceding numerators. Specifically, let $u_1 = a + b + c$, $u_2 = a + 3b + 3c$, and $u_3 = 3a + 5b + 9c$; then remaining numerators satisfy the recurrence $u_n = u_{n-1} + 2u_{n-2} + 4u_{n-3}$. The terms of Equation (3) are integers provided that each u_n is divisible by 2^n, and they are *even* integers if each u_n is divisible by 2^{n+1}. Thus, we seek to find a weighted tribonacci sequence (u_n) of positive integers that satisfies the infinite system of congruences:

$$u_n \equiv 0 \pmod{2^{n+1}} \qquad \text{for } n = 1, 2, 3, \ldots.$$

Unfortunately, there is no such sequence (u_n) that satisfies this infinite system of congruences, a result that follows from Theorem 6. The proof of Theorem 6 was shown to us by an anonymous user of MathOverflow [7] and involves concepts from both Galois theory and the p-adic numbers. Since these topics are somewhat beyond the scope of this article, we invite the reader unfamiliar with them to consider this proof as an invitation to explore these delightful areas of advanced algebra and number theory. Galois theory considers permutations of roots of polynomials such that any equation satisfied by the roots is still satisfied after the roots are permuted. We recommend the text by Cox as a nice introduction to Galois theory [2]. The p-adic numbers are naturally applicable to the problem at hand because they involve divisibility by prime powers. Importantly, for the prime $p = 2$, the 2-adic absolute value of an integer r is $|r|_2 = 2^{-v}$, where v is the largest integer such that 2^v divides r. Employed in the proof below, this absolute value is extended to the rationals and other field extensions. For a friendly introduction to the p-adics, we recommend the text by Gouvêa [3]. We now present the theorem.

Theorem 6. *Let (u_n) satisfy the recurrence*

$$u_n = u_{n-1} + 2u_{n-2} + 4u_{n-3}.$$

If (u_n) is not the zero sequence, then there is some maximal power 2^m that divides the terms (u_n).

Proof. Since the recurrence relation is linear with constant coefficients, we consider its characteristic polynomial $f(x) = x^3 - x^2 - 2x - 4$. This polynomial has three distinct roots, which we label α_1, α_2, α_3. These roots lie in $\overline{\mathbb{Q}}$, the algebraic closure of the rational numbers. The u_n can be expressed in terms of these roots:

$$u_n = A_1\alpha_1^n + A_2\alpha_2^n + A_3\alpha_3^n$$

for constants $A_1, A_2, A_3 \in \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ obtained from the initial conditions u_1, u_2, u_3. Specifically, we have the matrix equation

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 \\ \alpha_1^3 & \alpha_2^3 & \alpha_3^3 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Since matrix is a Vandermonde matrix and thus invertible, we can express the A_i in terms of the u_i.

Any Galois automorphism sending α_i to α_j also sends A_i to A_j. Consequently, if one of the $A_i = 0$, then all of the $A_i = 0$, and thus $u_n = 0$.

Instead, if $A_1 \neq 0$, we employ 2-adic numbers to complete the proof. Fix an embedding of $\overline{\mathbb{Q}}$ into the 2-adic field $\mathbf{Q}_2$. The Newton polygon of f consists of a line segment from $(0, 2)$ to $(2, 0)$ and another line segment from $(2, 0)$ to $(3, 0)$. This implies that f has exactly two roots with 2-adic valuation -1 and one root with 2-adic valuation 0. Let α_1 be the root with valuation 0, so α_2 and α_3 have valuation -1. Then the following holds, where $|\cdot|_2$ denotes the 2-adic absolute value:

$$|A_1\alpha_1^n|_2 = |A_1|_2, \qquad |A_2\alpha_2^n|_2 = |A_2|_2 \cdot 2^{-n}, \qquad |A_3\alpha_3^n|_2 = |A_3|_2 \cdot 2^{-n}.$$

For sufficiently large n,

$$|u_n|_2 = |A_1\alpha_1^n + A_2\alpha_2^n + A_3\alpha_3^n|_2 = |A_1|_2$$

by the ultrametric inequality (which is actually an equality since the terms inside the absolute value have different 2-adic valuations). Since $|u_n|_2 = |A_1|_2$ for sufficiently large n, the 2-adic valuation of u_n is eventually constant. This implies that there is some largest power of 2 that divides u_n. $\square$

With the result of Theorem 6, we now resolve the question of the existence of subprime tribonacci sequences consisting entirely of nonzero even terms.

Theorem 7. *The only subprime tribonacci sequence consisting entirely of even terms is the zero sequence.*

Proof. Suppose a, b, c are the initial terms of a subprime tribonacci sequence consisting entirely of even terms. The sequence can be written as in Equation (3), with the weighted tribonacci sequence of numerators (u_n) as defined previously.

If (u_n) is not the zero sequence, then by Theorem 6 there is a maximal power of 2 that divides the u_n. This means that Equation (3) is not a sequence of integers, which is a contradiction.

Thus, it must be that $u_n = 0$ for all n. Now a, b, and c can be obtained from the u_n by inverting the matrix in the following equation:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 5 & 9 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Since the $u_n = 0$, it follows that $a = b = c = 0$. Therefore, the only subprime tribonacci sequence consisting entirely of even numbers is the zero sequence. $\square$

While there do not exist subprime tribonacci sequence consisting entirely of nonzero even terms, there do exist subprime tribonacci sequences that start with arbitrarily many nonzero even terms. For a simple example, consider the sequence starting with $a = b = c = 2^m$ for some positive integer m. This sequence has at least $m + 2$ consecutive even terms; we leave the details to the reader.

Other examples can be obtained by finding nonzero a, b, c such that the u_n satisfy a finite system of congruences:

$$u_1 \equiv 0 \pmod{4}, \quad u_2 \equiv 0 \pmod{8}, \quad \ldots, \quad u_m \equiv 0 \pmod{2^{m+1}}.$$

4 Parting Thoughts

Having gained insight into subprime tribonacci sequences, we are left with many more questions. Most notably, do all subprime tribonacci sequences eventually enter a cycle? Are there other subprime tribonacci cycles besides those that we have found? Are there any tame cycles of length larger than 4? Are there any shortcuts to determining the cycle type from a given starting triple, other than computing a potentially long list of terms until a cycle (or a previously identified sequence) is detected?

We could also consider the consequences of modifying the definition of subprime tribonacci sequences. What if negative numbers were allowed in these sequences? What if we modify the recurrence relation to sum more than three consecutive terms? We could thus study subprime *tetranicci* sequences (involving sums of four terms), subprime *pentanacci* sequences (five terms), etc. What cycles emerge in these cases? How does the number or length of the cycles depend on the number of terms in the recurrence relation?

Computational investigation can help in answering these and other questions. As a starting point, our Python notebook, containing code that we used in our investigation of subprime tribonacci sequences, is available at `https://github.com/mlwright84/subtrib`. With basic coding ability and an eye for patterns, students can investigate the questions above and make their own discoveries. We look forward to seeing future results of this work!

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